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Elliptic bootstrapping and the nonlinear Cauchy–Riemann equation

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Abstract

The goal of this paper is to deduce a nonlinear elliptic regularity result from a linear one. In particular, elliptic bootstrapping is a powerful method to determine the regularity of a solution to a partial differential equation. We apply elliptic bootstrapping and linear elliptic regularity to the nonlinear Cauchy–Riemann equation. In doing so, we generalize the fundamental analytic result that holomorphic functions are automatically smooth. In particular, we show that, under certain conditions, the same is true for so-called J-holomorphic functions. We conclude by discussing how this nonlinear regularity result relates to ideas in symplectic geometry.

Suppose we have a C^k (i.e., k-times continuously differential) function $F : \mathbb{R}^n \to \mathbb{R}^n$. Suppose furthermore that we have a C^1-solution to the nonlinear ordinary differential equation

$$\dot{x} = F(x).$$

Roughly speaking, we see that x should have "one more derivative" than $F(x)$ via the following argument: Notice that $F(x) \in C^1$, so $\dot{x} \in C^1$ too. But this implies that $x \in C^2$. Thus $F(x)$ is actually in C^2, so that $\dot{x} \in C^2$ too. This implies that $x \in C^3$, and so on. We may continue this until we get that $F(x) \in C^k$, so $x \in C^{k+1}$. After this, even though $x \in C^{k+1}$, we cannot conclude that $F(x) \in C^{k+1}$ since F is only C^k. Thus we see that x is differentiable at least one more time than F is.

This is the essence of elliptic bootstrapping, namely by using the regularity of the coefficients of some differential equation in order to improve the regularity of any solution to that differential equation. (By "regularity," we simply mean "smoothness," or "how many times the function can be differentiated.")

Following the presentations in McDuff–Salamon [MS12, Appendix B] and Wendl [Wen15, Section 2.11], we give a more nontrivial example of elliptic bootstrapping.

In particular, one astonishing fact of complex analysis is that holomorphic functions are automatically smooth. Another way to phrase this is that "solutions to the Cauchy–Riemann equation are smooth." It turns out that this rests on a certain property of the Cauchy–Riemann equation known as *ellipticity*. While the general theory of elliptic partial differential equations is beyond the scope of this article, we will explore a generalization of the Cauchy–Riemann equation and prove via elliptic bootstrapping that its solutions are also automatically smooth.

One way to generalize holomorphic functions is to define a so-called *complex manifold*. An n-dimensional complex manifold is simply a $2n$-dimensional smooth manifold whose transition functions are holomorphic. Much as how we may talk about smooth functions on a smooth manifold, we may also talk about holomorphic functions on a complex manifold. A holomorphic function on a complex manifold is smooth: Locally, a complex manifold is exactly $\mathbb{C}^n$. But smoothness is a local condition, so the question of smoothness of holomorphic functions on a complex manifold reduces to the question of smoothness of holomorphic functions on $\mathbb{C}^n$.

There is, however, a further generalization of holomorphic functions to spaces known as *almost complex manifolds*. These manifolds arise naturally out of symplectic geometry, and they come with their own notions of holomorphic curves, often called *J-holomorphic* or *pseudoholomorphic* curves. These curves are solutions to the *nonlinear Cauchy–Riemann equation*, which generalizes the typical Cauchy–Riemann equations in complex analysis. We will prove via elliptic bootstrapping that, under relatively relaxed conditions, any J-holomorphic curve is automatically smooth.

We will discuss almost complex manifolds in Section 1. We will spend Section 2 introducing the Sobolev spaces $W^{k,p}$, which can be thought of as spaces of functions "admitting $k - n/p$ derivatives." In the end, using a bootstrapping argument, we will prove in Theorem 3.1 that, if the almost complex structure on an almost complex manifold is smooth, then any associated holomorphic curve is also smooth. In particular, we deduce a nonlinear elliptic regularity result from a linear one, which we state without proof. Finally, in Section 4, we will briefly and informally discuss the importance of this result in the context of symplectic geometry. We assume some familiarity with manifolds, multivariable calculus, and L^p-spaces. It would be helpful also to have seen some facts about complex analysis and partial differential equations.

1. *J*-holomorphic curves

An **almost complex structure** is a vector bundle homomorphism $J : TX \to TX$ such that $J^2 = -\operatorname{id}$ on the tangent spaces. We denote the set of C^ℓ-almost complex structures on a manifold X by $\mathcal{J}^\ell(X)$; if $\ell = \infty$, we also write $\mathcal{J}(X) := \mathcal{J}^\infty(X)$.

Example 1.1. Let $X = \mathbb{C}^n$ with coordinates $z_j = s_j + it_j$, and consider the standard complex structure J_0 on $\mathbb{C}^n$, which is defined on each tangent space $T_p\mathbb{C}^n = \mathbb{C}^n$ as

$$J_0\left(\frac{\partial}{\partial s_j}\bigg|_p\right) = \frac{\partial}{\partial t_j}\bigg|_p, \qquad J_0\left(\frac{\partial}{\partial t_j}\bigg|_p\right) = -\frac{\partial}{\partial s_j}\bigg|_p.$$

(From now on, we omit the subscript $|_p$, which only serves to denote which tangent space J_0 is acting on.) In other words, we may write J_0 in matrix form as

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. ♢

Many manifolds do not admit any almost complex structure at all. Indeed, we have the following proposition.

Proposition 1.2. *Suppose that X is an almost complex manifold, i.e., that it is a smooth manifold equipped with an almost complex structure J. Then X is even-dimensional and orientable.*

Proof. Say $\dim X = n$. If $p \in X$, then $J_p : T_pX \to T_pX$ is a vector space isomorphism between n-dimensional vector spaces such that $J_p^2 = -\operatorname{id}$, which has determinant $(-1)^n$. Thus $(-1)^n = (\det J_p)^2 \geq 0$, and so $n = 2k$ is even.

To show orientability, consider an arbitrary Riemannian metric h on X. Define $g(v,w) := h(v,w) + h(Jv,Jw)$, so that

$$g(Jv,Jw) = h(Jv,Jw) + h(J^2v,J^2w) = h(Jv,Jw) + (-1)^2h(v,w) = g(v,w).$$

Then define the $\omega(v,w) := g(v,Jw)$. Note that this is skew-symmetric since

$$\omega(w,v) = g(w,Jv) = g(Jw,J^2v) = -g(Jw,v) = -\omega(v,w)$$

by symmetry of g. On the other hand, we know that $\omega(v,-Jv) = g(v,v) \geq 0$, with equality if and only if $v = 0$. Thus ω is a nondegenerate 2-form. Then the k-th wedge product ω^k is a nowhere vanishing $2k$-form. But a nowhere vanishing top form defines an orientation, so we are done. □

Consider a compact two-dimensional smooth manifold Σ equipped with an almost complex structure j. (It turns out, in fact, that for this low-dimensional case, such a manifold is necessarily a *complex* manifold, in the sense that it admits coordinate charts with holomorphic transition functions [Don11, Theorem 22]. In general, however, almost complex does not imply complex, though the opposite is true.)

Our main object of study will be so-called *J-holomorphic curves* from (Σ, j) to the almost complex manifold (X, J). In particular, if $u \in C^\infty(\Sigma, X)$ satisfies

$$du \circ j = J \circ du,$$

then we call it a **J-holomorphic curve**.

We may now define an operator

$$\begin{aligned} \overline{\partial}_J : C^\infty(\Sigma, X) &\to \Omega^{0,1}(\Sigma, u^*TX) \\ u &\mapsto \frac{1}{2}(du + J \circ du \circ j) \end{aligned}$$

taking a smooth map $u : \Sigma \to X$ to a complex antilinear 1-form on Σ with values in the pullback tangent bundle

$$u^*TX = \{(p, v) : p \in \Sigma, v \in T_{u(p)}X\}.$$

By *complex antilinear*, we mean that it anticommutes with the almost complex strucutres; that is, we say $\omega \in \Omega^{0,1}(\Sigma, u^*TX)$ if $J \circ \omega = -\omega \circ j$. This operator $\overline{\partial}_J$ is often called the **del bar operator**. Then we have the following equivalent characterization of J-holomorphic curves.

LEMMA 1.3. *A smooth map $u : (\Sigma, j) \to (X, J)$ is J-holomorphic if and only if $\overline{\partial}_J(u) = 0$.*

Proof. Recall that $J^2 = -\operatorname{id}_{TX}$. Furthermore, we know that u is J-holomorphic if and only if $du \circ j = J \circ du$, which is in turn true if and only if $J \circ du - du \circ j = 0$. Now $-J$ is an isomorphism with inverse J, so $J \circ du - du \circ j = 0$ if and only if

$$du + J \circ du \circ -j = -J\left(J \circ du - du \circ j\right) = 0,$$

i.e., if and only if $\overline{\partial}_J u = 0$. This proves equivalence of our two definitions of J-holomorphic curves. □

Remark 1.4. To understand the $\overline{\partial}_J$ operator more explicitly, note that at each point $p \in \Sigma$, we have

$$\overline{\partial}_J(u)(p) = \frac{1}{2}\left(du_p + J_{u(p)} \circ du_p \circ j_p\right).$$

Now $du_p : T_p\Sigma \to T_{u(p)}X$; this codomain is exactly the fiber of u^*TX over the point $u(p) \in X$. On the other hand, we know that j_p is an endomorphism of $T_p\Sigma$, while $J_{u(p)}$ is an endomorphism of $T_{u(p)}X$. Thus $J_{u(p)} \circ du_p \circ j_p$ makes sense, and also maps from $T_p\Sigma$ to $T_{u(p)}X$. In particular, this is what it means to be a form "on Σ with values in u^*TX."

Now to verify that $\overline{\partial}_J$ does indeed take values in $\Omega^{0,1}(\Sigma, u^*TX)$, it suffices to show that

$$J \circ \overline{\partial}_J(u) = -\overline{\partial}_J(u) \circ j.$$

(This is, indeed, what it means to be a *complex antilinear form.*) But we see that

$$2J \circ \overline{\partial}_J(u) = J \circ du + J^2 \circ du \circ j = -J \circ du \circ j^2 - du \circ j = -2\overline{\partial}_J(u) \circ j,$$

where all we use is the fact that $J^2 = -\operatorname{id}_{TX}$ and $j^2 = -\operatorname{id}_{T\Sigma}$.

Example 1.5. Let $\{U_\alpha, \phi_\alpha\}$ be holomorphic coordinate charts on Σ. That is to say, the maps $\phi_\alpha : U_\alpha \to \mathbb{C}$ are diffeomorphisms such that $\phi_\alpha \circ \phi_\beta^{-1}$ are holomorphic maps of (open subsets of) $\mathbb{C}$. Recall that the almost complex structure on Σ is induced by these coordinate charts and the complex structure J_0 on $\mathbb{C}$. Then $u : (\Sigma, j) \to (X, J)$ is J-holomorphic if and only if each

$$u_\alpha := u \circ \phi_\alpha^{-1} : (\mathbb{C}, J_0) \supseteq (\phi_\alpha(U_\alpha), J_0) \to (X, J)$$

is J-holomorphic. Letting the coordinates of $\phi_\alpha(U_\alpha) \subseteq \mathbb{C}$ be $z = s + it$, we see that

$$\begin{aligned}\overline{\partial}_J u_\alpha &= \frac{1}{2}\left(du_\alpha + J \circ du_\alpha \circ J_0\right)\\ &= \frac{1}{2}\partial_s u_\alpha ds + \frac{1}{2}\partial_t u_\alpha dt - \frac{1}{2}J(u_\alpha)\partial_s u_\alpha ds \circ J_0 + \frac{1}{2}J(u_\alpha)\partial_t u_\alpha dt \circ J_0.\end{aligned}$$

Notice, however, that

$$(ds \circ J_0)\left(\frac{\partial}{\partial s}\right) = ds\left(\frac{\partial}{\partial t}\right) = 0, \qquad (ds \circ J_0)\left(\frac{\partial}{\partial t}\right) = ds\left(-\frac{\partial}{\partial s}\right) = -1.$$

Thus $ds \circ J_0 = -dt$. Similarly, we may check that $dt \circ J_0 = ds$. We find that

$$\overline{\partial}_J u_\alpha = \frac{1}{2}\left(\partial_s u_\alpha + J(u_\alpha)\partial_t u_\alpha\right) ds + \frac{1}{2}\left(\partial_t u_\alpha - J(u_\alpha)\partial_s u_\alpha\right) dt.$$

It follows that u_α is J-holomorphic if and only if $\partial_s u_\alpha + J(u_\alpha)\partial_t u_\alpha = 0$.

Now suppose that Σ and X are both simply $\mathbb{C}$ equipped with the standard holomorphic structure. Write $u = f + ig : \mathbb{C} \to \mathbb{C}$. Then the condition that u is J-holomorphic is exactly that

$$(\partial_s f + i\partial_s g) + J_0(\partial_t f + i\partial_t g) = (\partial_s f + i\partial_s g) + (i\partial_t f - \partial_t g) = 0.$$

In other words, a curve $u : \mathbb{C} \to \mathbb{C}$ is J_0-holomorphic exactly when it satisfies the Cauchy–Riemann equations

$$\partial_s f = \partial_t g, \qquad \partial_s g = -\partial_t f,$$

i.e., when it is holomorphic. Because of this, the operator $\overline{\partial}_J$ is often called the **nonlinear Cauchy–Riemann operator**. ◊

2. Sobolev spaces and weak equivalence

Our eventual goal is to have a statement of regularity for the nonlinear Cauchy–Riemann equation. However, this regularity result requires that we define a more general kind of space, known as a Sobolev space.

Loosely speaking, if $k \geq 0$ is an integer and $p \geq 1$ is a (possibly infinite) real number, then the **Sobolev space** $W^{k,p}(\Omega)$ on some open set $\Omega \subset \mathbb{R}^n$ is defined to be the set of L^p-functions u whose k-th derivatives exist and are also p-integrable. In this context, we often call Ω a **domain**.

While the above definition of $W^{k,p}(\Omega)$ is a helpful way of thinking about the space, it is not entirely accurate. In particular, we require only that a function in $W^{k,p}(\Omega)$ admit so-called *weak* derivatives, as opposed to the usual derivatives, which are accordingly known as *strong* derivatives.

Suppose $u : \Omega \to \mathbb{R}$ is a locally integrable function for a domain $\Omega \subset \mathbb{R}^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index of nonnegative integers α_i. Then the **α-th weak derivative** $D^\alpha u$ of u is a locally integrable function satisfying

$$\int_\Omega u(x)\partial^\alpha(\phi(x))\, dx = (-1)^{\alpha_1+\cdots+\alpha_n} \int_\Omega D^\alpha u(x)\phi(x)\, dx$$

for every compactly supported smooth function $\phi \in C_0^\infty(\Omega)$. Integration by parts implies that the above equation is always satisfied if u_α is the usual derivative. As such, this definition effectively asks that a weak derivative behave like the usual derivative under integration. Indeed, because integrals ignore what happens on a measure zero set, one may think of a weak derivative as a function which is the derivative almost everywhere.

Example 2.1. Let $\Omega = \mathbb{R}$, and define $u(x) = |x|$. This is locally integrable and admits the weak derivative

$$Du(x) = \begin{cases} -1 & \text{if } x < 0, \\ r & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Here r can be any real number. (In fact, any function which differs from the above formula for Du at a measure zero set is a weak derivative for u.) Furthermore, this weak derivative itself is p-integrable for any p, so that $u(x) \in W^{1,p}(\Omega)$. In fact, because u has further weak derivatives (namely functions which are 0 for $x \neq 0$), we actually have $u(x) \in W^{\infty,p}(\Omega)$. ♢

At this point, it is natural to wonder why we introduce the relatively complicated Sobolev spaces, rather than using C^k spaces, for example. The primary advantage is that Sobolev spaces are *complete*, which implies many theorems including Theorem 2.2 below. Indeed, we may define the **Sobolev**

norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p},$$

at least when $p \neq \infty$. Here the sum over $|\alpha| \le k$ indicates that we are summing over all multi-indices of length at most k. (When $p = \infty$, we may take the norm to be the maximum of the L^∞-norms of $D^\alpha u$, where α again ranges over all multi-indices of length at most k.) It turns out that this gives another way to define the Sobolev space $W^{k,p}(\Omega)$, namely as the completion of $C^\infty(\Omega)$ under the Sobolev norm $\|\cdot\|_{W^{k,p}(\Omega)}$, at least when $k \neq \infty$.

This definition of a Sobolev space generalizes to spaces of maps between manifolds, so that we may also define, for example, the space $W^{k,p}(\Sigma, X)$ to be the completion of $C^\infty(\Sigma, X)$ under the $W^{k,p}$-norm. For more information, one may look at [Wen15, pp. 126–128], for example.

To prove our regularity result, we will use a couple facts about Sobolev spaces.

Theorem 2.2 (Sobolev embedding theorem). *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded C^1 domain. If $kp > n$, then there is a continuous inclusion $W^{k,p}(\Omega) \hookrightarrow C^0(\Omega)$. If $kp < n$, then there is a continuous inclusion $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q = np/(n - kp)$.*

The proof of this is rather difficult, but we will simply take it for granted here. An interested reader may find it as Theorem 6 in [Eva10, Section 5.6.3]. As a note, it is actually enough to have Ω be a bounded Lipschitz domain; since we will mostly be working with balls, however, we may restrict our attention to C^1 domains. Furthermore, the Sobolev embedding theorem actually says more than what we have mentioned here. In particular, it shows that, for certain k and p, this is actually a *compact* inclusion. We will not require that fact, however.

Because Σ is two-dimensional, we will primarily work with domains in $\mathbb{C} = \mathbb{R}^2$; thus, when we apply the Sobolev embedding theorem, we will generally have $n = 2$. In this $n = 2$ case, we have the following corollary of Hölder's inequality, which gives us our first use of the Sobolev embedding theorem and will be used in the proof of Theorem 3.3.

Lemma 2.3 ([MS12, Lemma B.4.5]). *If $p > 2$ and $1 < r \le p$ and $\Omega \subset \mathbb{R}^2$ is any open set, then $f \in W^{1,p}(\Omega)$ and $g \in W^{1,r}(\Omega)$ together imply that $fg \in W^{1,r}(\Omega)$.*

Proof. It is enough to show that $D(fg) = f(Dg) + (Df)g \in L^r$ given that $Df \in L^p$ and $Dg \in L^r$. The Sobolev embedding theorem implies that that

$W^{1,p}(\mathbb{R}^2) \hookrightarrow C^0(\mathbb{R}^2)$, so that $f(Dg) \in C^0 \cdot L^r \subset L^r$. Thus it is sufficient to show that $(Df)g \in L^r$.

Since $1 < r \leq p$, there exists $q = pr/(p-r) \in (0, \infty]$ so that $1/p + 1/q = 1/r$. Now consider the following generalization of Hölder's inequality:

$$\|uv\|_{L^r} \leq \|u\|_{L^p} \|v\|_{L^q} .$$

In particular, it follows that

$$\|(Df)g\|_{L^r} \leq \|Df\|_{L^p} \|g\|_{L^q} .$$

Notice that $g, Dg \in W^{1,r}$ implies that $g, Dg \in W^{1,r'}$ for any $r' \leq r$. Thus without loss of generality $r < 2$, and so $g \in L^q$ by the Sobolev embedding theorem. Now since $p > 2$, we know that $q = pr/(p-r) < 2r/(2-r)$, and so $W^{1,r} \subset L^q$, proving the lemma. □

Before turning to the statement and proof of our elliptic regularity result for J-holomorphic curves, we return to and generalize the notion of a weak derivative. Recall that we asked a weak derivative to behave the same way as the usual derivative under integration. In general, we may call two functions **weakly equivalent** if they behave the same way under integration. That is to say, if $f, g \in L^1(\Omega)$ satisfy

$$\int_\Omega u\phi = \int_\Omega v\phi$$

for every compactly supported smooth function $\phi \in C_0^\infty(\Omega)$, then we call f and g weakly equivalent.

Finally, we take a moment here to standardize certain notation. In general, we will always use C_0^∞ to refer to compactly supported smooth functions, rather than simply functions which vanish near infinity. We sometimes call an element $\phi \in C_0^\infty$ a **test function**. Furthermore, when we say that Ω is a domain, we will always assume that Ω is a C^1 bounded open set in $\mathbb{R}^2$.

3. **Elliptic regularity**

With these results in mind, we are now ready to state and deduce the following elliptic regularity result for the nonlinear Cauchy–Riemann equation. We closely follow [MS12, Appendix B.4] here.

THEOREM 3.1 (Elliptic regularity, [MS12, Theorem B.4.1]). *Suppose $k \geq 2$ is an integer, and $p > 2$ is a real number. If $j \in \mathcal{J}(\Sigma)$, $J \in \mathcal{J}^k(X)$, and $u \in W^{1,p}(\Sigma, X)$ is J-holomorphic, then $u \in W^{k+1,p}(\Sigma, X)$. In particular, if J is smooth, then so too is any J-holomorphic curve u.*

Note that it is enough to prove this result locally, since being $W^{k+1,p}$ is a local condition. Thus, in this local setting, we may rephrase the theorem as follows: Suppose $\Omega \subseteq \mathbb{C}$ is open. Let J be a C^k-almost complex structure on $\mathbb{R}^{2n}$. (This J is obtained by pushing forward the original C^k-almost complex structure on X by a smooth local coordinate map.) Suppose furthermore that $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$ satisfies

$$\partial_s u + J(u)\partial_t u = 0.$$

Then $u \in W^{k+1,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$. (Notice that we use local integrability here, since u need not satisfy any particular constraints at the boundary of Ω.)

Note that J is a C^k-almost complex structure on $\mathbb{R}^{2n}$, where $k \geq 2$. Thus $J \circ u$ is a $W^{1,p}_{\text{loc}}$-almost complex structure on the domain of u, namely Ω. In particular, we have $J \circ u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n\times 2n})$. Note now that u is a $(J \circ u)$-holomorphic map. If we can use this to show that u was actually in $W^{2,p}_{\text{loc}}$, then we would have that $J \circ u$ is actually a $W^{2,p}_{\text{loc}}$-almost complex structure, and so on. We would be able to continue this process on until $W^{k,p}_{\text{loc}}$. This argument is known as **elliptic bootstrapping**, and is used often to improve the regularity of solutions to elliptic partial differential equations.

In particular, it would be enough to prove the following.

THEOREM 3.2. *Suppose $\Omega \subseteq \mathbb{C}$ is an open, bounded, C^1 domain and $J \in W^{k,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n\times 2n})$ satisfies $J^2 = -\mathbb{1}$. If $\partial_s u + J\partial_t u = 0$ then $u \in W^{k+1,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$.*

To prove this local version of elliptic regularity, we must first weaken our hypotheses somewhat. First, instead of requiring that $\partial_s u + J(u)\partial_t u = 0$, we must allow $\partial_s u + J(u)\partial_t u = \eta$ for some suitably regular $\eta : \Omega \to \mathbb{R}^{2n}$. Furthermore, we will actually want to consider $u \in L^q$ for some q, but the expression $\partial_s u$ is not well-defined in this case, since u is only integrable. Indeed, we want the notion, discussed in Section 2, of taking weak derivatives. For clarity we will explicitly state what weak equivalence means in this context.

If u had had first derivatives, then we would know by integration by parts that

$$(*) \qquad \int_\Omega \left\langle \partial_s \phi + J^T \partial_t \phi, u \right\rangle = -\int_\Omega \langle \phi, \partial_s u + \partial_t (Ju)\rangle = -\int_\Omega \langle \phi, \eta + (\partial_t J)u\rangle$$

for every test function $\phi \in C_0^\infty(\Omega, \mathbb{R}^{2n})$, where J^T denotes the transpose of $J : \Omega \to \text{End}(T\mathbb{R}^{2n}) = \mathbb{R}^{2n\times 2n}$. To see this equality, we use the fact that

$$\int_\Omega \langle \partial_s \phi, u\rangle = -\int_\Omega \langle \phi, \partial_s\rangle$$

by integration by parts, and that

$$\int_\Omega \left\langle J^T \partial_t \phi, u \right\rangle = \int_\Omega \langle \partial_t \phi, Ju\rangle$$

by definition of the transpose. Thus we will say that $\partial_s u + J\partial_t u = \eta$ **weakly** when Equation $(*)$ is satisfied.

We will prove the following proposition, which only assumes our weakened hypotheses.

PROPOSITION 3.3 ([MS12, Proposition B.4.9]). *Consider a bounded C^1 domain $\Omega \subset \mathbb{C}$. Let $J \in W^{\ell,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n\times 2n})$ satisfy $J^2 = -\mathbb{1}$, where ℓ is a positive integer and $p > 2$ is a real number. Suppose $u \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ and $\eta \in W^{\ell,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ satisfy $\partial_s u + J\partial_t u = \eta$ weakly, i.e., satisfy Equation (∗). Then $u \in W^{\ell+1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$, and $\partial_s u + J\partial_t u = \eta$ almost everywhere.*

This proposition proves Theorem 3.2, which in turn, as discussed earlier, proves our global statement of elliptic regularity in Theorem 3.1.

Proof of Theorem 3.2. Suppose $\partial_s u + J\partial_t u = 0$, where $J \in W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n\times 2n})$ satisfies $J^2 = -\mathbb{1}$. Notice that $\eta = 0$ is, in particular, an element of $W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ for every k. Now we apply Theorem 3.3 with $\eta = 0$ and $\ell = k$. Since $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ also belongs to L^p, it follows that $u \in W^{k+1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$, as desired. □

Before we can prove Theorem 3.3, however, we must prove the following statement. Its main purpose is that, when combined with the second part of the Sobolev embedding theorem, this theorem "upgrades" regularity (for certain q): Theorem 3.4 says an L^q function is actually $W^{1,r}$, while the Sobolev embedding theorem says that, under certain conditions, this $W^{1,r}$ function is actually $L^{q'}$ for some $q' > q$.

PROPOSITION 3.4 ([MS12, Proposition B.4.6]). *Let $\Omega \subset \mathbb{C}$ be a bounded C^1 domain. Suppose $p, q, r \in \mathbb{R}_+ \cup \{\infty\}$ such that*

$$2 < p, \qquad 1 < r < \infty, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Suppose further that $J \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n\times 2n})$ satisfies $J^2 = -\mathbb{1}$. Let $u \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ and $\eta \in L^r_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ satisfy

$$\int_\Omega \left\langle \partial_s \phi + J^T \partial_t \phi, u \right\rangle = \int_\Omega \langle \phi, \eta + (\partial_t J) u \rangle$$

for every $\phi \in C^\infty_0(\Omega, \mathbb{R}^{2n})$. Then $u \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ and $\partial_s u + J\partial_t u = \eta$ almost everywhere.

To prove this, we require one highly nontrivial fact, known as *linear* elliptic regularity. In particular, recall that the Laplacian Δu is simply $\partial^2 u/\partial s^2 + \partial^2 u/\partial t^2$. Then we have the following theorem.

THEOREM 3.5 (Linear elliptic regularity, [MS12, Theorem B.3.1]). *If Δu is weakly equivalent to $\partial_s f + \partial_t g$ for some $f, g \in L^r$, then $u \in W^{1,r}$.*

In some ways, this is the key fact which allows our proof of Theorem 3.1 to go through. We do not describe a proof here, but for some intuition for this fact, notice that $\partial_s f, \partial_t g$ have one fewer derivative than f and g do; of course, since $f, g \in L^r$, we think of them as having "zero derivatives," so we can roughly think of $\partial_s f, \partial_t g$ as elements of $W^{-1,r}$. Then $\Delta u \in W^{-1,r}$, and so u, which has two more derivatives than Δu, should belong to $W^{1,r}$.

Proof of Theorem 3.4. Let $\psi \in C_0^\infty(\Omega, \mathbb{R}^{2n})$ be arbitrary. Then set

$$\phi := \partial_s \psi - J^T \partial_t \psi \in W^{1,p}(\Omega, \mathbb{R}^{2n}).$$

This belongs to $W^{1,p}$ since J, hence its transpose J^T, does. Notice that Equation ($*$) is satisfied for $W^{1,p}$ functions, too, since smooth functions are dense in Sobolev spaces. In particular, recall our alternate definition of Sobolev spaces as completions of C^∞ under the Sobolev norm. As such, since Equation ($*$) behaves well under limits, we may consider $W^{1,p}$ functions as well.

In particular, Equation ($*$) is satisfied for this particular value of ϕ, even though ϕ is not actually smooth. We may compute that

$$\begin{aligned}
\partial_s \phi + J^T \partial_t \phi &= \partial_s^2 \psi - \partial_s \left(J^T \partial_t \psi \right) + J^T \partial_t \partial_s \psi - J^T \partial_t \left(J^T \partial_t \psi \right) \\
&= \partial_s^2 \psi - (\partial_s J^T)(\partial_t \psi) - J^T \partial_s \partial_t \psi + J_T \partial_t \partial_s \psi \\
&\quad - (J^T)^2 \partial_t^2 \psi - J^T (\partial_t J^T)(\partial_t \psi).
\end{aligned}$$

Notice that $(J^T)^2 = (J^2)^T = -\mathbb{1}$. Furthermore, because J^2 is constant, we know that

$$0 = \partial_t (J^2) = J \partial_t J + (\partial_t J) J.$$

The same holds when we take transposes, and so we conclude that

$$\begin{aligned}
\partial_s \phi + J^T \partial_t \phi &= \partial_s^2 \psi - (\partial_s J)^T (\partial_t \psi) + \partial_t^2 \psi + (\partial_s J)^T J^T (\partial_t \psi) \\
&= \Delta \psi - (\partial_s J)^T (\partial_t \psi) + (\partial_t J)^T J^T (\partial_t \psi).
\end{aligned}$$

In particular, we find that

$$\int_\Omega \langle \Delta \psi, u \rangle = \int_\Omega \left\langle \partial_s \phi + J^T \partial_t \phi, u \right\rangle - \int_\Omega \left\langle (\partial_t J)^T J^T \partial_t \psi, u \right\rangle + \int_\Omega \left\langle (\partial_s J)^T (\partial_t \psi), u \right\rangle.$$

Using the fact that u and η satisfy Equation ($*$), we know that this first integral is equal to

$$-\int_\Omega \langle \phi, \eta + (\partial_t J) u \rangle = -\int_\Omega \left\langle \partial_s \psi - J^T \partial_t \psi, \eta + (\partial_t J) u \right\rangle.$$

Rearranging so that the left-hand terms in each of the inner products is either $\partial_s\psi$ or $\partial_t\psi$, we see that

$$\int_\Omega \langle \Delta\psi, u\rangle = -\int_\Omega \langle \partial_s\psi, \eta + (\partial_t J)u\rangle + \int_\Omega \langle \partial_t\psi, J\eta + J\left((\partial_t J)u\right)\rangle$$
$$- \int_\Omega \langle \partial_t\psi, J\left((\partial_t J)u\right)\rangle + \int_\Omega \langle \partial_t\psi, (\partial_s J)u\rangle .$$

Setting $f := \eta + (\partial_t J)u$ and $g := -J\eta - (\partial_s J)u$, we now see that

$$\int_\Omega \langle \Delta\psi, u\rangle = -\int_\Omega \langle \partial_s\psi, f\rangle - \int_\Omega \langle \partial_t\psi, g\rangle .$$

At this point, we would like to use Theorem 3.5. By integration by parts, the above equation tells us that $\Delta u = \partial_s f + \partial_t g$ weakly, as they behave the same under integration. Now $\eta \in L^r_{\text{loc}}$ by hypothesis. Furthermore, since $J \in W^{1,p}_{\text{loc}}$, we know that $\partial_t J \in L^p_{\text{loc}}$. Since $u \in L^q_{\text{loc}}$, and $1/p + 1/q = 1/r$, we know that $(\partial_t J)u \in L^p_{\text{loc}} \cdot L^q_{\text{loc}} \subseteq L^r_{\text{loc}}$. Hence $f \in L^r_{\text{loc}}$. Similarly we may check that $g \in L^r_{\text{loc}}$.

Thus Δu is weakly equivalent to $\partial_s f + \partial_t g$ for $f, g \in L^r_{\text{loc}}$. Theorem 3.5 implies that $u \in W^{1,r}_{\text{loc}}$, as desired. □

We now prove that Theorem 3.4 implies Theorem 3.3.

Proof of Theorem 3.3. We break this proof into three steps: First, assuming $k = 1$, we will prove that $u \in W^{1,p}$. Second, we will prove that $u \in W^{2,p}$, which completes the $k = 1$ case. Finally, we will prove the general case.

STEP 1. $J \in W^{1,p}_{\text{loc}}$, $\eta \in W^{1,p}_{\text{loc}}$, $u \in L^p_{\text{loc}}$ *implies* $u \in W^{1,p}_{\text{loc}}$.

It is possible to find finite sequences $\{q_0, \ldots, q_m\}$ and $\{r_0, \ldots, r_m\}$ such that the following four conditions hold:

$$\frac{p}{p-1} < q_0 \le p, \qquad q_{m-1} < \frac{2p}{p-2} < q_m,$$
$$q_{j+1} := \frac{2r_j}{2-r_j}, \qquad r_j := \frac{pq_j}{p+q_j}.$$

In particular, we define r_j so that $1/p + 1/q_j = 1/r_j$; furthermore, the conditions in the first line guarantee that $r_j \neq 1, \infty$, so that Theorem 3.4 can be applied. Furthermore, we define q^{j+1} so that Theorem 2.2 holds. Finally, we may verify that the endpoints q_m, r_m are defined so that $r_m > 2$ and $r_0, \ldots, r_{m-1} < 2$.

In particular, notice that $1 < \frac{p}{p-1} < q_0 \le p$. Because $u \in L^p_{\text{loc}}$, it follows that $u \in L^{q_0}_{\text{loc}}$. Now by Theorem 3.4, we know that $u \in W^{1,r_0}_{\text{loc}}$. Because $r_0 < 2$, it follows by Theorem 2.2 that $u \in L^{q_1}_{\text{loc}}$ now.

Continuing in this fashion, we see that $u \in L^{q_m}_{\text{loc}}$, and so $u \in W^{1,r_m}_{\text{loc}}$. But now $r_m > 2$. By Theorem 2.2 again, we now have that $u \in C^0$, i.e., u is continuous.

But $C^0 \subset L^\infty_{\text{loc}}$, and so we now have that $u \in L^\infty_{\text{loc}}$. Furthermore, recall that $\eta \in W^{1,p}_{\text{loc}}$, so it certainly belongs to L^p_{loc} as well. Applying Theorem 3.4 with $q = \infty$ and $r = p$ now implies that $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$, as desired. (Recall that q may be ∞; only r must be finite.)

STEP 2. *$J \in W^{1,p}_{\text{loc}}$, $\eta \in W^{1,p}_{\text{loc}}$, $u \in W^{1,p}_{\text{loc}}$ implies $u \in W^{2,p}_{\text{loc}}$.*

We will begin by showing the following lemma.

LEMMA 3.6. *If $q, r > 1$ such that $1/p + 1/q = 1/r$, and if $u \in W^{1,q}_{\text{loc}}$, then $u \in W^{2,r}_{\text{loc}}$.*

Proof. Fix such q and r. Then set

$$\widetilde{u} := \partial_s u \in L^q_{\text{loc}}, \qquad \widetilde{\eta} := \partial_s \eta - (\partial_s J)\partial_t u.$$

Notice that $\partial_s \eta \in L^p_{\text{loc}} \subset L^r_{\text{loc}}$, where we use the fact that $p > r$. Furthermore, since $1/p + 1/q = 1/r$, we know that $(\partial_s J)\partial_t u \in L^p \cdot L^q_{\text{loc}} \subseteq L^r_{\text{loc}}$. Thus $\widetilde{\eta} \in L^r_{\text{loc}}$.

It turns out that $\widetilde{u}$ and $\widetilde{\eta}$ satisfy Equation $(*)$, in the sense that

$$\int_\Omega \left\langle \partial_s \phi + J^T \partial_t \phi, \widetilde{u} \right\rangle = -\int_\Omega \langle \phi, \partial_s \widetilde{u} + \partial_t (J\widetilde{u}) \rangle = -\int_\Omega \langle \phi, \widetilde{\eta} + (\partial_t J)\widetilde{u} \rangle$$

for all smooth test functions ϕ. To see this, observe that the first equality above follows directly from integration by parts. Thus it suffices to prove that

$$\partial_s \widetilde{u} + \partial_t (J\widetilde{u}) = \widetilde{\eta} + (\partial_t J)\widetilde{u}$$

weakly. But the left-hand side is exactly equal to

$$\partial_s \widetilde{u} + (\partial_t J)\widetilde{u} + J\partial_t \widetilde{u} = \partial_s^2 u + (\partial_t J)\partial_s u + J\partial_t \partial_s u.$$

On the other hand, using the definition for $\widetilde{\eta}$ and the hypothesis that $\partial_s u + J\partial_t u = \eta$ weakly, it follows that the right-hand side is given by

$$\partial_s (\partial_s u + J\partial_t u) - (\partial_s J)\partial_t u = \partial_s^2 u + (\partial_s J)\partial_t u + J\partial_s \partial_t u - (\partial_s J)\partial_t u = \partial_s^2 u + J\partial_s \partial_t u.$$

These two expressions are equal, since $\partial_s \partial_t u = \partial_t \partial_s u$. Thus $\widetilde{u}$ and $\widetilde{\eta}$ satisfy Equation $(*)$, as desired.

But now we may apply Theorem 3.4 to conclude that $\partial_s u = \widetilde{u} \in W^{1,r}$. If we could show that $\partial_t u \in W^{1,r}_{\text{loc}}$ as well, then we would have $u \in W^{2,r}_{\text{loc}}$, proving the fact. But notice that

$$\partial_t u = J(\partial_s u - \eta) \in W^{1,p}_{\text{loc}} \cdot W^{1,r}_{\text{loc}} \subseteq W^{1,r}_{\text{loc}},$$

where we use Theorem 2.3. This proves Theorem 3.6. □

Now the same argument using q_j and r_j from Step 1 holds. In particular, we eventually get that $u \in W^{2,r_j}_{\mathrm{loc}}$ for each j; since $r_m > 2$, it follows that u is continuously differentiable, and hence belongs to $W^{1,\infty}_{\mathrm{loc}}$. But now applying Theorem 3.6 with $q = \infty$ and $r = p$ implies that $u \in W^{2,p}$, as desired.

STEP 3. *$J \in W^{k,p}_{\mathrm{loc}}$, $\eta \in W^{k,p}_{\mathrm{loc}}$, $u \in L^p_{\mathrm{loc}}$ implies $u \in W^{k+1,p}_{\mathrm{loc}}$.*

We prove this inductively. In particular, suppose we have proven this step for some $k - 1 \geq 1$. Set $\widetilde{u}$ and $\widetilde{\eta}$ as before, so that they satisfy Equation ($*$) again. Then we find that $\partial_s u = \widetilde{u}$ and $\partial_t u$ are both in $W^{k-1,p}_{\mathrm{loc}}$, so that $u \in W^{k,p}_{\mathrm{loc}}$. This completes the induction. □

We showed earlier that Theorem 3.3 implies Theorem 3.2. As discussed toward the beginning of this section, Theorem 3.2 is a local statement of, and thus implies, our main regularity statement.

4. The moduli space of J-holomorphic curves

In this section, we discuss J-holomorphic curves in the context of symplectic geometry. This will be a relatively informal section; a small amount of algebraic topology (namely the notion of a fundamental class of a surface in homology) will be useful. We also briefly mention the first Chern class of a vector bundle, though it is only tangential to the larger story here.

A **symplectic form** ω on a smooth manifold X is a closed, nondegenerate 2-form. Being *closed* means that $d\omega = 0$, while being *nondegenerate* means that, for every nonzero tangent vector $v \in T_pX$, there exists $w \in T_pX$ so that $\omega_p(v, w) \neq 0$. If ω is a symplectic form on X, then we call (X, ω) a **symplectic manifold**. It turns out that any symplectic manifold has dimension $2n$, and ω^n is a nonvanishing top form, i.e., a volume form, on X. Hence X is orientable too.

Example 4.1. Consider the manifold $\mathbb{R}^{2n}$ (or $\mathbb{C}^n$). Define $\omega_{\mathrm{std}} := dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Recall that $d^2 = 0$, so

$$d\omega_{\mathrm{std}} = \sum_{i=1}^{n} (ddx_i \wedge dy_i - dx_i \wedge ddy_i) = 0.$$

Thus ω_{std} is closed. On the other hand, it is nondegenerate because

$$\omega_{\mathrm{std}}(p) \left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial y_i}\right|_p \right) = 1.$$

This is called the **standard symplectic structure**. In fact, Darboux's theorem says that every $2n$-dimensional symplectic manifold (X, ω) may be covered by coordinate charts in which the symplectic form may be written as $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. In particular, every symplectic manifold (X, ω)

is locally **symplectomorphic** to the standard symplectic manifold $(\mathbb{R}^{2n}, \omega_{\text{std}})$, in the sense that there are local diffeomorphisms ϕ between open sets of $\mathbb{R}^{2n}$ and X such that $\phi^*\omega = \omega_{\text{std}}$. $\Diamond$

Suppose now that J is an almost complex structure on X, i.e., is a map $J : TX \to TX$ with $J^2 = -\mathbb{1}$. If $\omega(v, Jv) > 0$ for every nonzero vector v and $\omega(v, w) = \omega(Jv, Jw)$ for every point $p \in X$ and every pair of vectors $v, w \in T_pX$, then we say that J is **ω-compatible**. The set of ω-compatible, C^ℓ-almost complex structures is written $\mathcal{J}^\ell(X, \omega)$. Furthermore, if $\ell = \infty$, then we omit the superscript.

Example 4.2. Recall the almost complex structure J_0 for $\mathbb{C}^n$ from Theorem 1.1. If $v = \sum_{i=1}^n \left(a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}\right)$ is a nonzero vector in $T_p\mathbb{R}^{2n}$, then we may compute

$$\begin{aligned}\omega(v, J_0 v) &= \omega\left(\sum_{i=1}^n \left(a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}\right), \sum_{i=1}^n \left(a_i \frac{\partial}{\partial y_i} - b_i \frac{\partial}{\partial x_i}\right)\right)\\ &= \sum_{i=1}^n \left(a_i^2 + b_i^2\right) > 0.\end{aligned}$$

(Another way to show $\omega(v, J_0 v) > 0$ for all nonzero v is to compute $\omega(v, J_0 v) = 1$ for all basis vectors v.) A similar computation shows that

$$\omega(v, w) = \omega(J_0 v, J_0 w),$$

and so J_0 is ω_{std}-compatible. $\Diamond$

Let (X, ω) be a symplectic manifold with compatible smooth almost complex structure $J \in \mathcal{J}(X, \omega)$. Let (Σ, j) be a compact two-dimensional almost complex manifold. For every homology class $A \in H_2(X; \mathbb{Z})$, define the space

$$\mathcal{M}(A, \Sigma; J) := \{u \in C^\infty(\Sigma, X) : [u] = A \text{ and } \overline{\partial}_J u = 0\}.$$

Here $[u]$ is simply the pushforward $u_*[\Sigma]$ of the fundamental class of Σ. We call this space the **moduli space** of J-holomorphic curves representing A. (The phrase “moduli space” simply means that this is a space whose points correspond to certain geometric objects—which, in this case, are J-holomorphic curves.)

We will, however, focus on a slightly simpler moduli space, namely the moduli space of all J-holomorphic maps representing A which are *simple*. In particular, say (Σ', j') is another compact two-dimensional almost complex manifold, and say $u' : (\Sigma', j') \to (X, J)$ is J-holomorphic. Suppose furthermore that there is a holomorphic branched covering $\phi : \Sigma \to \Sigma'$ so that $u' \circ \phi = u$. If, in this setting, we always have $\deg \phi = 1$, then we call u **simple**. A more geometric way to think about simple J-holomorphic maps is as maps which do not "cover their image multiple times." Then

$$\mathcal{M}^*(A, \Sigma; J) := \{u \in C^\infty(\Sigma, X) : [u] = A, \overline{\partial}_J u = 0, \text{ and } u \text{ is simple}\}$$

is the subset of $\mathcal{M}(A, \Sigma; J)$ consisting of simple J-holomorphic curves.

A priori, this moduli space has no manifold structure. Even if it were clearly a manifold, it is not clear that it would be finite-dimensional. It turns out, however, that we have the following theorem.

THEOREM 4.3 ([MS12, Theorem 3.1.6]). *For "generic" $J \in \mathcal{J}(X, \omega)$, the moduli space $\mathcal{M}^*(A, \Sigma; J)$ is a manifold of finite dimension.*

Remark 4.4. By *generic*, we mean that J belongs to a set $\mathcal{J}_{\text{reg}}(X, \omega) \subset \mathcal{J}(X, \omega)$ which contains an intersection of countably many open and dense subsets of $\mathcal{J}(X, \omega)$. Such a set is called **residual**. It is worth noting that, often, the "natural" choice of J is not actually generic, and work must be done in order to perturb J to be in this set $\mathcal{J}_{\text{reg}}(X, \omega)$. Certain regularity criteria are presented in [MS12, Section 3.3].

Remark 4.5. The theorem in [MS12] actually gives an exact formula for the dimension of this moduli space, namely $n(2 - 2g) + 2\langle c_1(TX), A\rangle$. Here g is the genus of Σ and $c_1(TX) \in H^2(X; \mathbb{Z})$ is the first Chern class. The inner product is the standard pairing between cohomology and homology.

The proof of this theorem turns out to depend somewhat heavily on Theorem 3.1. In particular, the theorem implies that, if $J \in \mathcal{J}^\ell$, then the space of $W^{k,p}$ J-holomorphic curves is independent of k, so long as $k \leq \ell + 1$. In particular, the space of J-holomorphic curves of class $W^{k,p}$ is independent of k whenever J is a smooth almost complex structure. This lets us work in $W^{k,p}$-neighborhoods when necessary; combined with completeness, this will allow us to show that $\mathcal{M}^*(A, \Sigma; J)$ is a finite-dimensional smooth submanifold of the space $W^{k,p}(\Sigma, X)$ of J-holomorphic curves $u : \Sigma \to X$ of class $W^{k,p}$.

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