Differential Forms in Algebraic Topology: Solutions

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Contents

Chapter 1

De Rham Theory

§1 The de Rham Complex on $\mathbb{R}^n$

> **Exercise** (p.17)**.** Show that the long exact sequence of cohomology groups exists and is exact.

Solution. We show that $\operatorname{im} f^* = \ker g^*$ and $\operatorname{im} g^* = \ker d^*$, and leave the proof that $\operatorname{im} d^* = \ker f^*$.

Because $gf = 0$, it is clear that $g^* f^* = 0$ as well, so that $\operatorname{im} f^* \subseteq \ker g^*$. On the other hand, suppose $g(b) = c$ can be written as $c = d_C(c')$ for some $c' \in C^{q-1}$, so that $[b] \in \ker g^*$. Pick some $b' \in B^{q-1}$ such that $g(b') = c'$. This is possible because g is surjective. Now we know that

$$g(d_B(b')) = d_C g(b') = c.$$

Thus $b - d_B(b') \in \ker g = \operatorname{im} f$. Write $a^q \in A^q$ to satisfy $f(a^q) = b - b^q$. Then notice that $f^*([a^q]) = [b + d_B(b')] = [b]$.

Now we prove $\operatorname{im} g^* = \ker d^*$. First notice that $\operatorname{im} g^* \subseteq \ker d^*$ because

$$d^* g^*([b]) = [f^{-1} db] = 0$$

since of course $db = 0$ if $[b] \in \ker d^*$. On the flip side, consider $c \in C^q$ be in $\ker d^*$. Thus, if we choose $b \in B^q$ so that $g(b) = c$, then $db = f(a)$ for some exact $a = d_A(a') \in A^{q+1}$. Let $b' = f(a') \in B^q$. Then we know that $g(b') = 0$, and so $g(b - b') = g(b) = c$. Furthermore, we know that

$$d_B(b - b') = db - db' = db - df(a') = db - f d_A(a') = 0.$$

Hence $b - b'$ represents a class in $H^q(B)$, and $g * ([b - b']) = [c]$, as desired. □

> **Exercise 1.7** (p.19)**.** Compute $H^*_{DR}(\mathbb{R}^2 - P - Q)$ where P and Q are two points in $\mathbb{R}^2$. Find the closed forms that represent the cohomology classes.

Solution. We know that an element of $H^0(\mathbb{R}^2 - P - Q)$ is just something in the kernel of $d : \Omega^0(\mathbb{R}^2 - P - Q) \to \Omega^1(\mathbb{R}^2 - P - Q)$, i.e., is a locally (hence globally) constant function f. Thus $H^0(\mathbb{R}^2 - P - Q) = \mathbb{R}$, where the cohomology class $x \in \mathbb{R}$ is represented by the constant function taking everything to x.

We use the integration trick to calculate H^1. Consider loops γ_1 and γ_2 winding once around P and Q, respectively, as seen in Figure 1. Then consider the map

$$\begin{aligned} \Phi : \ker(\Omega^1 \to \Omega^2) &\longrightarrow \mathbb{R}^2 \\ \omega &\longmapsto \left(\int_{\gamma_1} \omega, \int_{\gamma_2} \omega \right). \end{aligned}$$

If $\omega = df$ is exact, then $\int_{\gamma_i} df = \int_{\partial \gamma_i} f = 0$, and so $\omega \in \ker \Phi$. Hence Φ induces a map from H^1 to $\mathbb{R}^2$; we claim this map is an isomorphism. To do so, we must show, first, that $\ker \Phi$ consists precisely of the exact forms (we have only shown that exact forms are in $\ker \Phi$), and, second, that Φ is surjective.

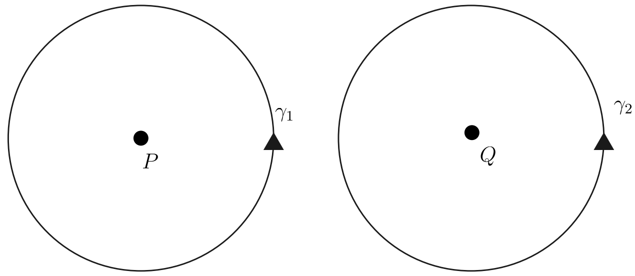


Figure 1: loops for integration

Suppose $\omega \in \ker \Phi$. We claim that $\int_\gamma \omega = 0$ for all loops γ. Certainly if γ doesn't go around P, Q, then it bounds a disk which doesn't contain either P or Q. Hence

$$\int_\gamma \omega = \int_{\text{disk}} d\omega = 0.$$

On the other hand, say γ bounds the disk D, and say $P \in D$. Say that γ_i bounds the disk C_i. Then we know that

$$\int_{\gamma-\gamma_1} \omega = \int_{D\setminus C_1} d\omega = 0.$$

But $\omega \in \ker \Phi$ implies that $\int_{\gamma_1} \omega = 0$, and so it follows that $\int_\gamma \omega = 0$ too. Of course, if $Q \in D$ (or if P and Q are *both* in D), the argument is the same.

Now we want to show that $\omega = df$ for some f. Consider

$$f(x) = \int_{\substack{\gamma \text{ from} \\ 0 \text{ to } x}} \omega.$$

This is well-defined: Suppose γ and γ' were loops from 0 to x. Then the integral over $\gamma - \gamma'$ is an integral of ω over some loop, which is thus zero, as desired.

We can write ω as

$$\omega_{(x,y)} = g_1(x, y)dx_{(x,y)} + g_2(x, y)dy_{(x,y)}.$$

(We think of $dy_{(x,y)}$ as "dy_p.") Then we can rewrite

$$f(x, y) = \int_0^x g_1(u, 0)\ du + \int_0^y g_2(x, v)\ dv = \int_0^y g_2(0, v)\ dv + \int_0^x g_1(u, y)\ du.$$

Using the second expression for f, we see that

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\int_0^x g_1(u, y)\ du\right) = g_1(x, y).$$

Similarly, using the first expression, we see that $\frac{\partial f}{\partial y} = g_2(x, y)$. Hence

$$df_{(x,y)} = g_1(x, y)\ dx_{(x,y)} + g_2(x, y)\ dy_{(x,y)} = \omega_{(x,y)},$$

as desired. This concludes the proof that $\ker \Phi = \{\text{exact 1-forms}\}$.

Now we would like to show that $\operatorname{im} \Phi = \mathbb{R}^2$. It is enough to find closed ω such that $\int_{\gamma_1} \omega = 0$ and $\int_{\gamma_2} \omega = 1$. We will use the argument principle: Consider some function f. Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\ dz = \sum \text{winding \# of } C \text{ around zeros} - \sum \text{winding \# of } C \text{ around poles.}$$

Thus, setting $\omega = \frac{f'(z)}{f(z)}\ dz$, we see that we want f to have a zero inside γ_1 and no zeros or poles inside γ_2. Of course, some function like $f(z) = z - (\text{something in } \gamma_1)$ works! Hence $H^1 = \mathbb{R}^2$, as desired.

We claim $H^2 = 0$. This proof is somewhat annoying, especially given the tools we are given at this stage, so we only sketch a solution: First, consider the case of only one puncture, i.e., $H^2(\mathbb{R}^2 - \{0\})$. We would like to find an $\omega \in \Omega^1$ such that $d\omega = f(x, y)\ dxdy = f(r, \theta)\ drd\theta$. To do this, just let ω be obtained by integration along radial lines:

$$\omega = \left(\int_1^r f(t, \theta)\ dt\right) d\theta.$$

Now, for the case of two punctures, we have 1-forms ω_1 and ω_2 which look like the ω defined above near P and Q, respectively. Smoothing things out using a bump function (or perhaps a truncation function) gives the desired 1-form. □

§2 The Mayer–Vietoris Sequence

> **Exercise 2.1.1** (p.20). Show that if $\omega = \sum g_I du_I$, then $d\omega = \sum dg_I du_I$.

Solution. By linearity, it is enough to show this for $\omega = gdu_{i_1} \dots du_{i_q}$. But then we know that

$$\begin{aligned} d\omega &= d(gdu_{i_1} \dots du_{i_q}) \\ &= d\left(\sum_{j_1,\dots,j_q} g\frac{\partial u_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial u_{i_q}}{\partial x_{j_q}} dx_{j_1} \dots dx_{j_q}\right) \\ &= \sum_{j,j_1,\dots,j_q} \left(\left(\frac{\partial g}{\partial x_j}\frac{\partial u_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial u_{i_q}}{\partial x_{j_q}} dx_j\right) + \left(g\frac{\partial^2 u_{i_1}}{\partial x_j \partial x_{j_1}}\frac{\partial u_{i_2}}{\partial x_{i_2}} \cdots \frac{\partial u_{i_q}}{\partial x_{i_q}} dx_j\right) + \dots\right) dx_{j_1} \dots dx_{j_q}. \end{aligned}$$

On the other hand, we know that

$$\begin{aligned} dgdu_I &= \sum \frac{\partial g}{\partial x_j} dx_j du_{i_1} \dots du_{i_q} \\ &= \sum_{j,j_1,\dots,j_q} \frac{\partial g}{\partial x_j}\frac{\partial u_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial u_{i_q}}{\partial x_{j_q}} dx_j dx_{j_1} \dots dx_{j_q}. \end{aligned}$$

Thus it is enough to show that

$$\sum_{j,j_1,\dots,j_q} \left(\left(g\frac{\partial^2 u_{i_1}}{\partial x_j \partial x_{j_1}}\frac{\partial u_{i_2}}{\partial x_{j_2}} \cdots \frac{\partial u_{i_q}}{\partial x_{j_q}}\right) + \left(g\frac{\partial u_{i_1}}{\partial x_{j_1}}\frac{\partial^2 u_{i_2}}{\partial x_j \partial x_{j_2}} \cdots \frac{\partial u_{i_q}}{\partial x_{i_q}}\right) + \dots\right) dx_j dx_{j_1} \dots dx_{j_q} = 0.$$

This is true for the same reason as in Proposition 1.4. □

§3 Orientation and Integration

> **Exercise 3.1** (p.28). Show that $dT_1 \dots dT_n = J(T) dy_1 \dots dy_n$, where $J(T) = \det(\partial x_i / \partial y_j)$ is the Jacobian determinant of T.

Solution. Recall that $T_i = x_i$. Then

$$\begin{aligned} dT_1 \dots dT_n &= \sum_{i_1,\dots,i_n} \frac{\partial T_1}{\partial y_{i_1}} dy_{i_1} \cdots \frac{\partial T_n}{\partial y_{i_n}} dy_{i_n} \\ &= \sum_{i_1,\dots,i_n} \frac{\partial x_1}{\partial y_{i_1}} \cdots \frac{\partial x_n}{\partial y_{i_n}} dy_{i_1} \dots dy_{i_n} \\ &= \sum \operatorname{sgn}(\pi) \frac{\partial x_1}{\partial y_{\pi(1)}} \cdots \frac{\partial x_n}{\partial y_{\pi(n)}} dy_1 \dots dy_n, \end{aligned}$$

where π is the permutation taking y_{i_1} to y_1, and so on up to y_{i_n} to y_n. This last sum, however, is exactly the determinant of the Jacobian matrix! □

> **Exercise 3.6** (p.33)**.** Prove Stokes's theorem for the upper half space.

Solution. Now ω is an $(n-1)$-form:

$$\omega = f_n dx_1 \dots dx_{n-1} + \cdots + f_1 dx_2 \dots dx_n.$$

Note that each f_i is a function $f_i(x_1, \dots, x_n)$. Then

$$d\omega = \left(\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} + \cdots + (-1)^{n-1} \frac{\partial f_n}{\partial x_n} \right) dx_1 \dots dx_n.$$

Say the condition for the upper half space is that $x_n \geq 0$. Then we know that

$$\int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x_i} dx_1 \dots dx_n = \int_0^\infty \left(\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{\partial f_i}{\partial x_i} dx_1 \dots dx_{n-1} \right) dx_n.$$

If $i = 1, \dots, n-1$, then we can rearrange the integrals within the parentheses so that the innermost one is

$$\int_{-\infty}^\infty \frac{\partial f_i}{\partial x_i} dx_i = f_i(x_1, \dots, x_{i-1}, \infty, x_{i+1}, \dots, f_n) - f_i(x_1, \dots, x_{i-1}, -\infty, x_{i+1}, \dots, f_n) = 0,$$

since f_i is compactly supported. It follows that

$$\int_{\mathbb{H}^n} d\omega = (-1)^{n-1} \left(\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty -f_n(x_1, \dots, x_{n-1}, 0) dx_1 \dots dx_{n-1} \right) = \int_{\partial \mathbb{H}^n} \omega,$$

where we use the fact that $\mathbb{H}^n$ has the induced orientationgiven by the equivalence class of the form $(-1)^n dx_1 \dots dx_{n-1}$. □

§4 Poincaré Lemmas

> **Exercise 4.2** (p.36)**.** Show that $r : \mathbb{R}^2 - \{0\} \to S^1$ given by $r(x) = x/\|x\|$ is a deformation retraction.

Solution. The straight-line homotopy works. □

> **Exercise 4.3** (p.36)**.** Cover S^n by two open sets U and V where U is slightly larger than the northern hemisphere and V slightly larger than the southern hemisphere. Then $U \cap V$ is diffeomorphic to $S^{n-1} \times \mathbb{R}^1$ where S^{n-1} is the equator. Using the Mayer–Vietoris sequence, show that
>
> $$H^*(S^n) = \begin{cases} \mathbb{R} & \text{in dimensions } 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Notice that $U \simeq \mathbb{R}^n \simeq V$, while $U \cap V \simeq S^{n-1} \times \mathbb{R}^1$ is homotopy equivalent to S^{n-1}. An argument similar to the one for Exercise 1.7 shows that $H^*(\mathbb{R}^2 - \{0\})$ is $\mathbb{R}$ when $* = 0, 1$ and 0 otherwise. We may thus induct and suppose that we already know the de Rham cohomology of S^{n-1}.

Now we have the following piece of the Mayer–Vietoris sequence:

	S^n	$U \amalg V = \mathbb{R}^n \amalg \mathbb{R}^n$	$U \cap V = S^{n-1}$
$n+1$	0 →	0 →	$\dots$
n	$\mathbb{R}$ →	0 →	0
$n-1$	0 →	0 →	$\mathbb{R}$
$n-2$	$\dots$ →	0 →	0

For all $i > n$, we know that $H^i(S^n) = 0$ because the terms around it are both zero. The same holds for i between 2 and $n-1$. It then follows that we have an isomorphism between $H^{n-1}(S^{n-1})$ and $H^n(S^n)$, so $H^n(S^n) = \mathbb{R}$ as desired.

On the bottom, we have another piece of the Mayer–Vietoris sequence:

$$
\begin{array}{cccccc}
2 & \to 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
1 & \to ? & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & ? & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R}
\end{array}
$$

(Note that there is implicitly a row of 0's below.) It is enough to know what the map $\Omega^0(U) \oplus \Omega^0(V) \to \Omega^0(U \cap V)$ is. It takes (ω, τ) to $\tau - \omega$. We showed before that this is surjective: If $\omega \in \Omega^0(U \cap V)$ is closed and if $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$, then $\xi = (-\rho_v \omega, \rho_U \omega)$ maps to ω. Hence $H^0(S^n) = \mathbb{R}$ while $H^1(S^n) = 0$, which completes the proof. □

Exercise 4.3.1 (p.37)**.** Let $S^n(r)$ be the sphere of radius r

$$x_1^2 + \cdots + x_{n+1}^2 = r^2$$

in $\mathbb{R}^{n+1}$, and let

$$\omega = \frac{1}{r} \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \dots \widehat{dx_i} \dots dx_{n+1}.$$

(a) Write S^n for the unit sphere $S^n(1)$. Compute the integral $\int_{S^n} \omega$ and conclude that ω is not exact.

(b) Regarding r as a function on $\mathbb{R}^{n+1} - 0$, show that $(dr) \cdot \omega = dx_1 \dots dx_{n+1}$. Thus ω is the Euclidean volume form on the sphere $S^n(r)$.

Solution.

(a) Certainly if $\int_{S^n} \omega = 0$, then ω could not be exact by Stokes's theorem. In particular, if $\omega = d\eta$ then $\int_{S^n} \omega = \int_{\partial S^n} \eta = 0$ since $\partial S^n = \emptyset$.

However, we know that

$$\int_{S^n} \omega = \sum_{i=1}^{n+1} \int_{S^n} (-1)^{i-1} x_i dx_1 \dots \widehat{dx_i} \dots dx_{n+1}.$$

We know, however, that

$$\int_{S^n} (-1)^{i-1} x_i dx_1 \dots \widehat{dx_i} \dots dx_{n+1} = \int_{B^{n+1}} dx_1 \dots dx_{n+1},$$

which is just the volume of the $(n+1)$-dimensional ball, and so

$$\int_{S^n} \omega = \sum_{i=1}^{n+1} \operatorname{vol}(B^{n+1}) = (n+1) \operatorname{vol}(B^{n+1}) \neq 0,$$

as desired.

(b) We know that $dr = \sum \frac{\partial r}{\partial x_i} dx_i$. It thus follows that

$$\begin{aligned} dr \cdot \omega &= \left(\sum \frac{\partial r}{\partial x_i} dx_i \right) \cdot \frac{1}{r} \sum (-1)^{i-1} x_i dx_1 \dots \widehat{dx_i} \dots dx_{n+1} \\ &= \frac{1}{r} \sum_{i=1}^{n+1} \frac{\partial r}{\partial x_i} x_i dx_1 \dots dx_{n+1}. \end{aligned}$$

We can evaluate explicitly, however, that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, where again $r = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$. But then, using dV as shorthand for $dx_1 \dots dx_{n+1}$, we see that

$$dr \cdot \omega = \frac{1}{r}\sum_{i=1}^{n+1} \frac{x_i^2}{r} dV = dV,$$

since of course $r^2 = \sum x_i^2$. □

Exercise 4.5 (p.38)**.** Show that $d\pi_* = \pi_* d$; in other words, $\pi_* : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M)$ is a chain map.

Solution. First, we show that this is true on type (I) forms. On one hand, we know that $d\pi_* = 0$. On the other hand, we calculate the following:

$$\begin{aligned}\pi_* d(\pi^*\phi \cdot f(x,t)) &= \pi^*\left(\pi^* d\phi \cdot f(x,t) + \pi^*\phi\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt\right)\right) \\ &= \pi_*\left(\pi^*\phi \frac{\partial f}{\partial t}dt\right) \\ &= \phi\int_{-\infty}^{\infty} \frac{\partial f}{\partial t}dt \\ &= \phi[f(x,\infty) - f(x,-\infty)].\end{aligned}$$

Both terms are 0 because f is compactly supported, which completes the proof for type (I) forms.

For type (II) forms, we have

$$\begin{aligned}d\pi_*(\pi^*\phi \cdot f(x,t)dt) &= d\left(\phi\int_{-\infty}^{\infty} f(x,t)dt\right) \\ &= d\phi\int_{-\infty}^{\infty} f(x,t)dt + \phi\frac{\partial[\int_{-\infty}^{\infty} f(x,t)dt]}{\partial x}dxdt.\end{aligned}$$

The Leibniz integral rule implies that

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty} f(x,t)dt = \int_{-\infty}^{\infty}\frac{\partial f(x,t)}{\partial x}dt,$$

so that this expression is in fact equal to

$$\begin{aligned}\pi_* d(\pi^*\phi \cdot f(x,t)dt) &= \pi_*\left(\pi^* d\phi \cdot f(x,t)dt + \pi^*\phi\frac{\partial f}{\partial x}dxdt\right) \\ &= d\phi\int_{-\infty}^{\infty} f(x,t)dt + \phi dx\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}dt.\end{aligned}$$

□

Exercise 4.8 (p.40)**.** Compute the cohomology groups $H^*(M)$ and $H_c^*(M)$ of the open Möbius strip M, i.e., the Möbius strip without the bounding edge. [*Hint:* Apply the Mayer–Vietoris sequences.]

Solution. Because M is two-dimensional, we only need to worry about H^i for i up to 2.

We break up the Möbius strip as shown in the

It is fairly quick to see that $H^2(M) = 0$. On the other hand, to find H^1 and H^0, we must understand the map $\delta : \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R}$ from $H^0(U \amalg V) = H^0(\mathbb{R}^2 \amalg \mathbb{R}^2)$ to $H^0(U \cap V) = H^0(\mathbb{R}^2 \amalg \mathbb{R}^2)$. But $\delta(\omega, \tau) = (\omega - \tau, \omega - \tau)$, which is clearly one-dimensional. It follows at this point that

$$H^*(M) = \begin{cases}\mathbb{R} & \text{if } * = 0, 1, \\ 0 & \text{otherwise.}\end{cases}$$

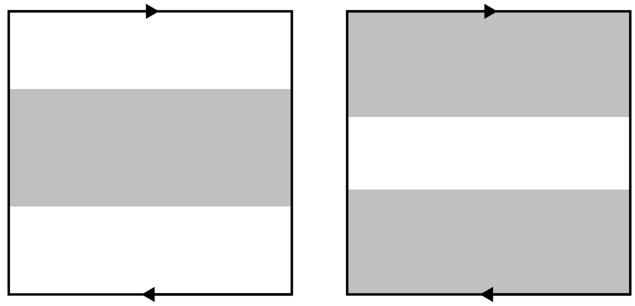


Figure 2: A cover $U \cup V$ of the Möbius strip.

Now for $H_c^*(M)$, we have the following Mayer–Vietoris sequence:

$$\begin{array}{cccccc}
 & M & & U \amalg V & & U \cap V \\
H^3 & \cdots & \longleftarrow & 0 & \longleftarrow & 0 \\
H^2 & ? & \longleftarrow & \mathbb{R} \oplus \mathbb{R} & \longleftarrow & \mathbb{R} \oplus \mathbb{R} \\
H^1 & ? & \longleftarrow & 0 & \longleftarrow & 0 \\
H^0 & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

The map $\mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$ takes $\omega = (\omega_1, \omega_2)$ to $(-(j_U)_*\omega, (j_V)_*\omega)$. We check that

$$(j_U)_*\omega_1 = \omega_1 \qquad (j_V) * \omega_1 = \omega_1$$
$$(j_U)_*\omega_2 = \omega_2 \qquad (j_V) * \omega_2 = -\omega_2,$$

and so $\partial(\omega) = (-\omega_1 - \omega_2, \omega_1 - \omega_2)$. In particular, im ∂ is two-dimensional, and so ∂ is an isomorphism. Hence $H_c^*(M) = 0$. □

> **Exercise 4.10.1** (p.41)**.** Prove that the image of a proper map is closed.

Solution. Say $f : X \to Y$ is a proper map. Pick $p \in Y \setminus f(X)$ and pick a neighborhood N of p such that $\bar{N}$ is compact. Then $f^{-1}(\bar{N})$ is compact while $f^{-1}(N)$ is open. Now $f(f^{-1}(\bar{N}))$ is compact and contained in $\bar{N}$. Note that this doesn't contain p because $p \notin \operatorname{im} f$.

Thus consider the nonempty set $\bar{N} - f(f^{-1}(\bar{N}))$. This is an open neighborhood containing p. Furthermore, its intersection with $\operatorname{im} f$ is necessarily empty: If $f(x) \in \bar{N} - f(f^{-1}(\bar{N}))$, then $x \in f^{-1}(\bar{N})$ and so $f(x) \in f(f^{-1}(\bar{N}))$. Hence $\operatorname{im} f$ is closed, as desired. □

> **Exercise 4.11.1** (p.42)**.** Prove Theorem 4.11 (Sard's theorem) from Sard's theorem for $\mathbb{R}^n$.

Solution. Say $f : M \to N$. Every manifold has a countable atlas, so we may consider a countable atlas $\{W_i\}$ for N. We know that $f^{-1}(W_i)$ is open in M, and so each $f^{-1}(W_i)$ can be trivialized by countably many coordinate charts. (For example, letting $\{U_i\}$ be a countable atlas of M, let $U_{ij} = f^{-1}(W_i) \cap U_j$.) Thus we now have a countable atlas $\{U_{ij}\}$ of M, where $f(U_{ij}) \subset W_i$.

Let S be the set of critical points of f and let ϕ_{ij} be the coordinate map associated to U_{ij}. We would like to show that $\phi_{ij}(S \cap U_{ij})$ has measure zero for each pair (i, j). But we know that $p \in S \cap U_{ij}$ if and only if $df_p : T_pM \to T_{f(p)}N$ is not surjective, which is in turn true if and only if $d(\psi_i \circ f \circ \phi_{ij}^{-1})_{\phi_{ij}(p)}$ is not surjective. After all, we know that $d\psi_i$ and $d\phi_{ij}$ are both bijective since coordinate maps are diffeomorphisms. (Here, ψ_i denotes the coordinate map associated to $W_i \subset N$.) We conclude by Sard's theorem for $\mathbb{R}^n$ that the critical points S of $f : M \to N$ forms a measure zero subset. □

§5 The Mayer–Vietoris Argument

Exercise 5.5 (p.44)**.** Prove the Five Lemma: Given a commutative diagram of abelian groups and group homomorphisms

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D & \xrightarrow{f_4} & E & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} & & \\ \cdots & \longrightarrow & A' & \xrightarrow[f_1']{} & B' & \xrightarrow[f_2']{} & C' & \xrightarrow[f_3']{} & D' & \xrightarrow[f_4']{} & E' & \longrightarrow & \cdots \end{array}$$

in which the rows are exact, if the maps α, β, δ, and ε are isomorphisms, then so is the middle one γ.

Solution. This is just a diagram chase. □

Exercise 5.12 (p.50)**.** The Künneth formula for compact cohomology states that for any manifolds M and N having a finite good cover,

$$H_c^*(M \times N) = H_c^*(M) \otimes H_c^*(N).$$

(a) In the case M and N are orientable, show that this is a consequence of Poincaré duality and the Künneth formula for de Rham cohomology.

(b) Using the Mayer–Vietoris argument prove the Künneth formula for compact cohomology for any M and N having a finite good cover.

Solution.

(a) Let $\dim M = m$ and $\dim N = n$. Then, for any p and q, we know by Poincaré duality that $H_c^{m-p}(M) = H^p(M)$ and that $H_c^{n-q}(N) = H^q(N)$, while $H_c^{m+n-p-q}(M \times N) = H^{p+q}(M \times N)$. The conclusion follows from the Künneth formula.

(b) Let $U \cup V = M$. Then we have the Mayer–Vietoris sequence:

$$\cdots \to H_c^p(U \cap V) \to H_c^p(U) \oplus H^p(V) \to H_c^p(U \cup V) \to \dots.$$

We tensor with $H_c^{n-p}(N)$, where n is some fixed integer, to get another exact sequence. Summing over all p gives us

$$\begin{aligned} \dots &\to \bigoplus_{p=0}^{n} H_c^p(U \cap V) \otimes H_c^{n-p}(N) \\ &\to \left[\bigoplus_{p=0}^{n} H_c^p(U) \otimes H_c^{n-p}(N)\right] \oplus \left[\bigoplus_{p=0}^{n} H_c^p(V) \otimes H_c^{n-p}(N)\right] \\ &\to \bigoplus_{p=0}^{n} H_c^p(U \cup V) \otimes H_c^{n-p}(N) \to \dots. \end{aligned}$$

Now we have the commutative diagram

$$\begin{array}{ccccccc} \dots \to \bigoplus_{p=0}^{n} H_c^p(U \cap V) \otimes H_c^{n-p}(N) & \longrightarrow & \text{stuff} & \longrightarrow & \bigoplus_{p=0}^{n} H_c^p(U \cup V) \otimes H_c^{n-p}(N) \to \dots \\ \downarrow{\scriptstyle\psi} & & \downarrow & & \downarrow \\ \dots \longrightarrow H^n((U \cap V) \times N) & \longrightarrow & H^n(U \times N) \oplus H^n(V \times N) & \longrightarrow & H^n((U \cup V) \times N) \longrightarrow \dots \end{array}$$

(I didn't copy in the stuff in the center, just because it wouldn't fit, and also would make the diagram unreadable.) Here, the downward maps are all given by $\psi : H^*(M) \otimes H^*(N) \to H^*(M \times N)$, which is the map sending $\omega \otimes \phi \mapsto \pi^*\omega \wedge \rho^*\phi$ for projections $\pi : M \times N \to M$ and $\rho : M \times N \to N$.

First, we show that the square

$$\begin{array}{ccc} \displaystyle\bigoplus_{p=0}^{n} H_c^p(U\cap V)\otimes H_c^{n-p}(N) & \longrightarrow & \text{stuff} \\ \psi\downarrow & & \downarrow\psi \\ H^n((U\cap V)\times N) & \longrightarrow & H^n(U\times N)\oplus H^n(V\times N) \end{array}$$

commutes. To see this, fix a summand $\omega\otimes\phi\in H_c^p(U\cap V)\otimes H_c^{n-p}(N)$ in the top left corner. Taking the bottom path, we see that it is taken to

$$(-(j_U)_*[\pi^*\omega\wedge\rho^*\phi],(j_V)_*[\pi^*\omega\wedge\rho^*\phi])\in H^n(U\times N)\oplus H^n(V\times N).$$

On the other hand, taking the bottom path, we see that the first arrow sends $\omega\otimes\phi$ to

$$(-(j_U)_*(\omega\otimes\phi),(j_V)_*(\omega\otimes\phi)).$$

Now we know that

$$\psi(-(j_U)_*(\omega\otimes\phi))=-\pi^*(j_U)_*\omega\wedge\rho^*\phi.$$

But $(j_U)*\omega$ is just ω on U, and extends by 0 from $U\cap V$. In particular, this is exactly $-(j_U)_*[\pi^*\omega\wedge\rho^*\phi]$. The other term follows similarly.

Commutativity of the other two squares is clear. □

Exercise 5.16 (p.52)**.** Let x, y be the standard coordinates and r,θ the polar coordinates on $\mathbb{R}^2-\{0\}$.

(a) Show that the Poincaré dual of the ray $\{(x,0):x>0\}$ in $\mathbb{R}^2-\{0\}$ is $d\theta/2\pi$ in $H^1(\mathbb{R}^2-\{0\})$.

(b) Show that the closed Poincaré dual of the unit circle in $H^1(\mathbb{R}^2-\{0\})$ is 0, but the compact Poincaré dual is the nontrivial generator $\rho(r)dr$ in $H_c^1(\mathbb{R}^2-\{0\})$ where $\rho(r)$ is a bump function with total integral 1. (By a bump function we mean a smooth function whose support is contained in some disc and whose graph looks like a "bump.")

Thus the generator of $H^1(\mathbb{R}^2-\{0\})$ is represented by the ray and the generator of $H_c^1(\mathbb{R}^2-\{0\})$ by the circle.

Solution.

(a) We would like to show that

$$\int_{\text{ray}} i^*\omega=\int_{\mathbb{R}^2-\{0\}}\omega\wedge\frac{d\theta}{2\pi}$$

for every closed ω. Write $\omega=fdr+gd\theta$. Then $d\omega=0$ implies that $\frac{\partial f}{\partial\theta}=\frac{\partial g}{\partial r}$.

Now we know that

$$\frac{d}{d\theta}\left[\int_0^\infty f(r,\theta)\ dr\right]=\int_0^\infty\frac{\partial f}{\partial\theta}dr=\int_0^\infty\frac{\partial g}{\partial r}dr=g(\infty,\theta)-g(0,\theta)=0$$

because ω is compactly supported. Hence $\int_0^\infty f(r,\theta)dr$ is constant with respect to θ. But then it follows that

$$2\pi\cdot\int_{\mathbb{R}^2-\{0\}}\omega\wedge\frac{d\theta}{2\pi}=\int_{\mathbb{R}^2-\{0\}}f(r,\theta)\ drd\theta=\int_0^{2\pi}\int_0^\infty f(r,\theta)drd\theta=2\pi\cdot\int_0^\infty f(r,0)dr.$$

But of course

$$\int_{\text{ray}}i^*\omega=\int_0^\infty f(r,0)dr,$$

concluding the proof.

(b) Again let $\omega = fdr + gd\theta$ be a closed form. Then we know that

$$\int_{S^1} i^*\omega = \int_0^{2\pi} g(1,\theta)d\theta.$$

We would like to show that this is equal to $\int \omega \wedge 0 = 0$. But we know that

$$\frac{d}{dr}\int_0^{2\pi} g(r,\theta)d\theta = \int_0^{2\pi} \frac{\partial g}{\partial r}d\theta = \int_0^{2\pi} \frac{\partial f}{\partial \theta}d\theta = f(r,2\pi) - f(r,0).$$

But $(r, 2\pi)$ and $(r, 0)$ are the same point, so this is 0. It follows, then, that $\int_0^{2\pi} g(1,\theta)d\theta$ is a constant. We may take the limit to either $r = 0$ or $r = \infty$. Compact support implies that this limit is 0, which concludes the proof.

To find the compact Poincaré dual, note that we would like to show that

$$\int_0^{2\pi} g(1,\theta)\ d\theta = \int_{S^1} i^*\omega = \int_{\mathbb{R}^2-\{0\}} \omega \wedge \rho(r)dr = \int_{\mathbb{R}^2-\{0\}} g(r,\theta)\rho(r)d\theta dr.$$

But we know from the argument above that this last term is equal to

$$\int_0^\infty \int_0^{2\pi} g(r,\theta)\rho(r)d\theta dr = \left[\int_0^\infty \rho(r)dr\right] \cdot \left[\int_0^{2\pi} g(1,\theta)d\theta\right].$$

The first term is 1, which proves the result. □

§6 The Thom Isomorphism

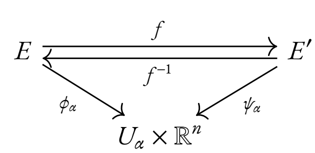
Exercise 6.2 (p.54)**.** Show that two vector bundles on M are isomorphic if and only if their cocycles relative to some open cover are equivalent.

Solution. We prove the forwards direction first. Suppose we have equivalent vector bundles $\pi : E \to M$ and $\pi' : E' \to M$. We may certainly pick an open cover $\{U_\alpha\}$ of M so that $\pi|_{U_\alpha}$ and $\pi'|_{U_\alpha}$ are both fiber-preserving diffeomorphisms. (To see this, simply take trivializing open covers $\{V_\alpha\}$ and $\{V'_\alpha\}$ for E and E', respectively, then let the open cover in question be obtained by the set of all possible $V_\alpha \cap V'_\beta$'s.)

Let $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ be the transition functions for the two vector bundles. Then

$$g_{\alpha\beta} = (\phi_\alpha f^{-1}\psi_\alpha^{-1}) \circ (\psi_\alpha\psi_\beta^{-1}) \circ (\psi_\beta f \phi_\beta^{-1}),$$

where the maps in question are depicted below:



Thus, defining $\lambda_\alpha = \phi_\alpha f^{-1}\psi_\alpha^{-1}$ works.

Now we prove the backwards direction, and let $\{U_\alpha\}$ be the open cover relative to which the two vector bundles are equivalent. On $E|_{U_\alpha}$, we define $f : E \to E'$ to be $\psi_\alpha^{-1}\lambda_\alpha\phi_\alpha$. This glues to a map on all of E: After all, on $\pi^{-1}(U_\alpha \cap U_\beta)$, we know that

$$\psi_\alpha^{-1}\lambda_\alpha\phi_\alpha = \psi_\alpha^{-1}(g_{\alpha\beta}\lambda_\beta g'_{\alpha\beta})\phi_\alpha = \psi_\beta^{-1}\lambda_\beta\phi_\beta,$$

as desired. This is also clearly fiber-preserving, hence a homomorphism. But we may similarly define a global function f^{-1} : $E' \to E$ to be $\phi_\alpha^{-1}\lambda_\alpha\psi_\alpha$ on each $E'|_{U_\alpha}$. Since f and f^{-1} are inverses, it follows that E and E' are actually isomorphic as vector bundles. □

Exercise 6.5 (p.56)**.**

(a) Show that there is a direct product decomposition

$$GL(n, \mathbb{R}) = O(n) \times \{\text{positive definite symmetric matrices}\}.$$

(b) Use (a) to show that the structure group of any real vector bundle may be reduced to $O(n)$ by finding the λ_α's of Lemma 6.1.

Solution.

(a) This is just the polar decomposition of a matrix.

(b)

TODO: I'm not sure how to prove this, actually.

□

Exercise 6.10 (p.59)**.** Compute $\mathrm{Vect}_k(S^1)$.

Solution. Note that any vector bundle over S^1 is given by some $\phi : \mathbb{R}^k \to \mathbb{R}^k$, i.e., by some $\phi \in GL(\mathbb{R}^k)$. In particular, a vector bundle over S^1 is just a (necessarily trivial) vector bundle over $[0, 1]$ such that $\pi^{-1}(0) = \mathbb{R}^k$ and $\pi^{-1}(1) = \mathbb{R}^k$ are glued together in some nontrivial way. Then we claim that $\phi_0, \phi_1 \in GL(\mathbb{R}^k)$ give the same vector bundle only when they are in the same path component, so that $\mathrm{Vect}_k(S^1)$ can be identified with the path components of $GL(\mathbb{R}^k)$.

Note that what we really have is a pullback bundle $f^{-1}E$ over the closed interval I, where $f : I \to S^1$ takes $t \mapsto e^{2\pi i t}$:

$$\begin{array}{ccc} I \times \mathbb{R}^k = f^{-1}E & \xrightarrow{F} & E \\ \downarrow{\scriptstyle \text{project}} & & \downarrow{\scriptstyle \pi} \\ I & \xrightarrow[f]{} & S^1 \end{array}$$

We know that $F(0, v) = F(1, \phi(v))$ for some $\phi \in GL(\mathbb{R}^k)$, and that F is an isomorphism on the other fibers.

Now suppose there is a path ϕ_t from ϕ_0 to ϕ_1. Let E_0 and E_1 be the vector bundles obtained from ϕ_0 and ϕ_1, respectively. Then consider the map $E_0 \to E_1$ given by $(t, \vec{v}) \mapsto (t, \phi_t \circ \phi_0^{-1}(\vec{v}))$. It is easy to see that this respects the gluing: At $t = 0$, this is the identity, while at $t = 1$, it takes $(1, \phi_0(\vec{v})) = (0, \vec{v})$ to $(1, \phi_1(\vec{v})) = (0, \vec{v})$. It follows that E_0 and E_1 are vector bundle isomorphic.

On the other hand, suppose there is a vector bundle isomorphism between E_0 and E_1. Ignoring the fiber over $0 = 1 \in S^1$, these vector bundles are both trivial. Thus an isomorphism between E_0 and E_1 is necessarily some map $(t, \vec{v}) \mapsto (t, \phi(\vec{v}))$ such that $(0, \phi(\vec{v})) = (1, \phi(\phi_0(\vec{v})))$ in E_1. This automatically puts E_1 into the desired form, and taking ϕ_t to be the map such that this isomorphism is $\phi_t \circ \phi_0^{-1}$ on the fiber over t gives a path between ϕ_0 and ϕ_1. □

Exercise 6.14 (p.62)**.** Show that if E is an oriented vector bundle, then $\pi_*\omega_\alpha = \pi_*\omega_\beta$. Hence $\{\pi_*\omega_\alpha\}_{\alpha\in I}$ piece together to give a global form $\pi_*\omega$ on M. Furthermore, this definition is independent of the choice of the oriented trivialization for E.

Solution. Certainly if ω is type (I) near a point, then $\pi_*\omega_\alpha = 0 = \pi_*\omega_\beta$. Suppose we have a change of coordinates $T \in GL^+(\mathbb{R}^{m+n})$ taking $(x_1, \dots, x_m, t_1, \dots, t_n)$ to $(y_1, \dots, y_m, u_1, \dots, u_n)$. In particular, we have $T^*\omega_\beta = \omega_\alpha$ and $T^*\pi^*\tau = \pi^*\phi$. Then

$$\int_{\mathbb{R}^n} g(y, u)du = \int_{\mathbb{R}^n} T^*(g(y, u)du) = \int_{\mathbb{R}^n} f(x, t)dt$$

because T is orientation-preserving. Since applying the change of coordinates to $\pi_*\omega_\beta$ turns the τ into a ϕ, we conclude that $\pi_*\omega_\alpha = \pi_*\omega_\beta$, as desired.

That this definition does not depend on the choice of trivialization follows from the same logic, this time instead using ω_α defined on U_α and ω_β defined on V_β. Note that the change of coordinates is still orientation-preserving since E was an oriented vector bundle. □

> **Exercise 6.20** (p.65)**.** Using a Mayer–Vietoris argument as in the proof of the Rhom isomorphism (Theorem 6.17), show that if $\pi : E \longrightarrow M$ is an orientable rank n bundle over a manifold M of finite type, then
>
> $$H_c^*(E) \simeq H_c^{*-n}(M).$$
>
> Note that this is Proposition 6.13 with the orientability assumption on M removed.

Solution. By the same argument as in Theorem 6.17, it is sufficient to show that

$$\begin{array}{ccccccccc}
\dots \longleftarrow & H_c^{*+1}(E|_{U\cap V}) & \xleftarrow{d_*} & H_c^{*+1}(E|_{U\cup V}) & \longleftarrow & H_c^*(E|_U) \oplus H_c^*(E|_V) & \longleftarrow & H_c^*(E|_{U\cap V}) & \longleftarrow \dots \\
& \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & \\
\dots \longleftarrow & H_c^{*-n+1}(U \cap V) & \xleftarrow[d_*]{} & H_c^{*-n}(U \cup V) & \longleftarrow & H_c^{*-n}(U) \oplus H_c^{*-n}(V) & \longleftarrow & H_c^{*-n}(U \cap V) & \longleftarrow \dots
\end{array}$$

commutes.

The only difficult part is checking the left square. The form $d_*\pi_*(\omega)$ satisfies the property that its extension to U is $d(\rho_U\pi_*(\omega))$. On the other hand, we know that $\pi_*d_*\omega$'s extension to U is obtained by first extending $d_*\omega$ to $E|_U$, and then projecting down. The extension of $d_*\omega$ is just $-d(\pi^*\rho_U\pi_*\omega) = -\pi^*(d\rho_U\pi_*\omega)$. It follows that $\pi_*d_*\omega$ extended to U is

$$-\pi_*((\pi^*d\rho_U) \wedge \omega) = d\rho_U\pi_*\omega.$$

□

> **Exercise 6.32** (p.70)**.**
>
> (a) Show that if θ is the standard angle function on $\mathbb{R}^2$, measured in the counterclockwise direction, then $d\theta$ is positive on the circle S^1.
>
> (b) Show that if ϕ and θ are the spherical coordinates on $\mathbb{R}^3$ as in Figure 6.7, then $d\phi \wedge d\theta$ is positive on S^2.

Solution.

(a) We would like to show that $dr \wedge \pi^*d\theta = drd\theta$ is positive on $\mathbb{R}^2 - \{0\}$. We write everything in xy-coordinates:

$$drd\theta = (2xdx + 2ydy) \wedge \frac{1}{1 + y^2/x^2}\left(\frac{dy}{x} - -\frac{ydx}{x^2}\right) = \left(\frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2}\right) dxdy.$$

This is indeed positive.

(b) Writing standard Euclidean coordinates x, y, z in terms of these spherical coordinates, we see that

$$\begin{aligned}
dx &= \sin\theta\cos\phi\ dx + r\cos\theta\cos\phi\ d\theta - r\sin\theta\sin\phi\ d\phi \\
dy &= \sin\theta\sin\phi\ dx + r\cos\theta\sin\phi\ d\theta + r\sin\theta\cos\phi\ d\phi \\
dx &= \cos\theta\ dx - r\sin\theta\ d\theta.
\end{aligned}$$

It follows then that

$$dxdydz = r^2(\sin\theta)\ dxd\theta d\phi.$$

But $r^2\sin\theta > 0$ because $\theta \in [0, \pi]$, and so $d\theta d\phi$ is positive. □

> **Exercise 6.36** (p.72)**.** There exist 1-forms ξ_α on U_α such that
>
> $$\frac{1}{2\pi}d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha.$$
>
> [*Hint:* Take $\xi_\alpha = (1/2\pi)\sum_\gamma \rho_\gamma d\varphi_{\gamma_\alpha}$, where $\{\rho_\gamma\}$ is a partition of unity subordinate to $\{U_\gamma\}$.]

Solution. Consider a partition of unity, as suggested. We want to show that

$$d\varphi_{\alpha\beta} = \sum_\gamma \rho_\gamma (d\varphi_{\gamma\beta} - d\varphi_{\gamma\alpha}).$$

Since $d\varphi_{\alpha\beta} + d\varphi_{\beta\gamma} - d\varphi_{\alpha\beta} = 0$, this is indeed true. □

> **Exercise 6.43** (p.75). Let $\pi : E \to M$ be an oriented rank 2 bundle. As we saw in the proof of the Thom isomorphism, wedging with the Thom class is an isomorphism $\wedge\Phi : H^*(M) \to H_{cv}^{*+2}(E)$. Therefore every cohomology class on E is the wedge product of Φ with the pullback of a cohomology class on M. Find the class u on M such that
>
> $$\Phi^2 = \Phi \wedge \pi^* u \text{ in } H_{cv}^*(E).$$

Solution. We want u so that $\mathcal{T}(u) = \pi^* u \wedge \Phi = -\Phi^2$, i.e., so that $-u = \pi_*(\Phi \wedge \Phi)$.

Recall (Equation 6.42) the explicit form of the Thom class. Thus $\Phi \wedge \Phi$ consists of three terms:

1. There is a
$$d\left(\rho(r)\frac{d\theta_\alpha}{2\pi}\right)^2$$
term. This is equal to
$$\left(\frac{\rho'(r)}{2\pi}\, drd\theta_\alpha\right)^2 = \left(\frac{\rho'(r)}{2\pi}\right)^2\, drd\theta_\alpha drd\theta_\alpha.$$
But $(drd\theta_\alpha)^2 = -(dr)^2(d\theta_\alpha)^2 = 0$.

2. There are two
$$d\left(\rho(r)\frac{d\theta_\alpha}{2\pi}\right) \wedge \frac{1}{2\pi i} d\left(\rho(r)\pi^* \sum_\gamma \rho_\gamma d(\log g_{\gamma\alpha})\right)$$
terms. Each such term is equal to
$$\frac{\rho'(r)}{2\pi} drd\theta_\alpha \wedge \frac{1}{2\pi i}\left[\rho'(r)dr\pi^* \sum_\gamma \rho_\gamma d(\log g_{\gamma\alpha}) + \rho(r)\pi^* \sum_\gamma d\rho_\gamma d(\log g_{\gamma\alpha})\right],$$
which is in turn equal to
$$\frac{1}{4\pi^2 i}\rho'(r)drd\theta_\alpha \wedge \rho(r)\pi^* \sum_\gamma d\rho_\gamma d(\log g_{\gamma\alpha}).$$
In U_α with local coordinates $x_1, \ldots, x_n$, we have that the above expression is just
$$\frac{1}{4\pi^2 i}\rho'(r)\rho(r) \sum_\gamma \left[\sum_{i,j} \frac{\partial \rho_\gamma}{\partial x_i} \frac{\partial(\log g_{\gamma\alpha})}{\partial x_j} dx_i dx_j\right] drd\theta_\alpha,$$
where we think of x_i as the coordinates on both U_α and on $E|_{U_\alpha}$. (Technically, we should apply π^* to the sum, so that entire sum portion of the expression acts as the $\pi^*\phi$ part.) Applying π_* and summing the two terms up gives us
$$\frac{1}{2\pi^2 i} \sum d\rho_\gamma d(\log g_{\gamma\alpha}) \int_0^{2\pi}\int_0^\infty \rho(r)\rho'(r)\, drd\theta.$$
Note that the double integral just evaluates to $-\pi$, so this ends up just being
$$-\frac{1}{2\pi i}\sum_\gamma d\rho_\gamma d(\log(g_{\gamma\alpha})).$$

3. Finally, there is a
$$-\frac{1}{4\pi^2}\left[\rho'(r)dr\pi^*\sum_\gamma \rho\gamma d\log g_{\gamma\alpha}+\rho(r)\pi^*\sum d\rho_\gamma d\log_{\gamma\alpha}\right]^2$$
term. Note, however, that there is no θ anywhere, so this term gets killed by π_*.

Thus u is just equal to the Euler class $e(E)$! □

Exercise 6.44 (p.75)**.** The complex projective space $\mathbb{C}P^n$ is the space of all lines through the origin in $\mathbb{C}^{n+1}$, topologized as the quotient of $\mathbb{C}^{n+1}$ by the equivalence relation
$$z\sim\lambda z \text{ for } z\in\mathbb{C}^{n+1},\quad \lambda \text{ a nonzero complex number.}$$
Let $z_0,\dots,z_n$ be the complex coordinates on $\mathbb{C}^{n+1}$. These give a set of homogeneous coordinates $[z_0,\dots,z_n]$ on $\mathbb{C}P^n$, determined up to multiplication by a nonzero complex number λ. Define U_i to be the open subset of $\mathbb{C}P^n$ given by $z_i\neq 0$. $\{U_0,\dots,U_n\}$ is called the standard open cover of $\mathbb{C}P^n$.

(a) Show that $\mathbb{C}P^n$ is a manifold.

(b) Find the transition functions of the normal bundle $N_{\mathbb{C}P^1/\mathbb{C}P^2}$, relative to the standard open cover of $\mathbb{C}P^1$.

Solution.

(a) I skipped this first part, just because it's pretty standard. Maybe the only hard part is to check that it is locally Euclidean, but one can just parametrize by slopes.

(b) The standard open cover of $\mathbb{C}P^1$ is given by the open sets $V_0=\{[1,z_1,0]\}$ and $V_1=\{[z_0,1,0]\}$, with trivializations ϕ_i given by z_i. Over $V_0=U_0\cap\mathbb{C}P^1$, we have affine coordinates $(z_1/z_0,z_2/z_0)$. Thus $\pi^{-1}(V_0)$ is spanned by $\frac{\partial}{\partial(z_2/z_0)}$. Similarly, the fibers over V_1 are spanned by $\frac{\partial}{\partial(z_2/z_1)}$.

At a point $[z_0,z_1,0]$, the transition function g_{01} should should thus satisfy
$$\frac{\partial}{\partial(z_2/z_0)}=g_{01}\frac{\partial}{\partial(z_2/z_1)},$$
i.e., we should have
$$g_{01}=\frac{\partial(z_2/z_1)}{\partial(z_2/z_0)}=\frac{\partial}{\partial x}\frac{z_1x}{z_0}=\frac{z_1}{z_0},$$
where $x=z_2/z_0$. □

Exercise 6.45 (p.77)**.** On the complex projective space $\mathbb{C}P^n$ there is a tautological line bundle S, called the universal subbundle; it is the subbundle of the product bundle $\mathbb{C}P^n\times\mathbb{C}^{n+1}$ given by
$$S=\{(\ell,z):z\in\ell\}.$$
Abo ve each point ℓ in $\mathbb{C}P^n$, the fiber of S is the line represented by ℓ. Find the transition functions of the universal subbundle S of $\mathbb{C}P^1$ relative to the standard open cover and compute its Euler class.

Solution. We use the standard open cover $\{U_0,U_1\}$ of $\mathbb{C}P^1$. Pick some $[z_0,z_1]\in U_0\cap U_1$. Its fiber in $S|_{U_0}$ is given by $\{\lambda\cdot(1,z_1)\}$, and similarly for its fiber in $S|_{U_1}$. Thus g_{01} takes
$$\left(\frac{z_0}{z_1},\lambda\right)\mapsto\left(\left[\frac{z_0}{z_1},1\right],\lambda\left(\frac{z_0}{z_1},1\right)\right)=\left([1,z_1/z_0],\frac{z_0\lambda}{z_1}(1,z_1/z_0)\right)\mapsto\left(\frac{z_1}{z_0},\frac{z_0\lambda}{z_1}\right).$$
Thus $g_{01}=z_0/z_1$. This is the coordinate on U_1, which we denote w. For convenience, we will denote the coordinate on U_0 as z. Note that $w=1/z$ on $U_0\cap U_1$.

Now the Euler class is
$$e(S) = -\frac{1}{2\pi i}d(\rho_0 d(\log g_{01})) = -\frac{1}{2\pi i}d(\rho_0 d\log w)$$
on U_1, where ρ_0 is 1 near the origin and 0 near infinity.

The same argument as in Exercise 6.44.1 shows that the integral $\int_{\mathbb{C}P^1} e(S) = -1$, where the negative comes from the fact that we use w instead of z this time. □

Exercise 6.46 (p.77)**.** Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and i the antipodal map on S^n:
$$i : (x_1, \dots, x_{n+1}) \to (-x_1, \dots, -x_{n+1}).$$
The real projective space $\mathbb{R}P^n$ is the quotient of S^n by the equivalence relation
$$x \sim i(x) \text{ for } x \in \mathbb{R}^{n+1}.$$

(a) An invariant form on S^n is a form ω such that $i^*\omega = \omega$. The vector space of invariant forms on S^n, denoted $\Omega^*(S^n)^I$, is a differential complex, and so the invariant cohomology $H^*(S^n)^I$ of S^n is defined. Show $H^*(\mathbb{R}P^n) \simeq H^*(S^n)^I$.

(b) Show that the natural map $H^*(S^n)^I \to H^*(S^n)$ is injective. [*Hint:* If ω is an invariant form and $\omega = d\tau$ for some form τ on S^n, then $\omega = d(\tau + i^*\tau)/2$.]

(c) Give S^n its standard orientation. Show that the antipodal map $i : S^n \to S^n$ is orientation-preserving for n odd and orientation-reversing for n even. Hence, if $[\sigma]$ is a generator of $H^n(S^n)$, then $[\sigma]$ is a nontrivial invariant cohomology class if and only if n is odd.

(d) Show that the de Rham cohomology of $\mathbb{R}P^n$ is
$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{R} & \text{for } q = 0, \\ 0 & \text{for } 0 < q < n, \\ \mathbb{R} & \text{for } q = n \text{ odd}, \\ 0 & \text{for } q = n \text{ even}. \end{cases}$$

Solution.

(a) Let $[\omega] \in H^*(\mathbb{R}P^n)$. Write $\omega = \sum f_I dx_I$ locally. Then, letting $\pi : S^n \to \mathbb{R}P^n$ be the quotient map $x \mapsto \{x, i(x)\}$, we have
$$\pi^*\left(\sum (f_i \circ i) dx_I\right) = \pi^* \sum f_i dx_I$$
since $i(x) = x$. This gives us a well-defined map $\pi^* : H^*(\mathbb{R}P^n) \to H^*(S^n)^I$.

Furthermore, it has inverse given by taking $\omega \in H^*(S^n)^I$ to the form η satisfying $\eta_{\{x,i(x)\}} = \omega_x$.

(b) The hint more or less gives it to us: If $\omega = d\tau$ then
$$\frac{d\tau}{2} + \frac{di^*\tau}{2} = \frac{\omega}{2} + \frac{\omega}{2} = \omega.$$
Note that $(\tau + i^*\tau)/2$ is invariant. This shows that $\ker(\Omega^*(S^n)^I \to H^*(S^n))$ is comprises just coboundaries.

(c) A map which swaps the sign in one coordinate is orientation-reversing. If n is odd, then i is obtained by composing an even number of orientation-reversing maps, and thus preserves orientation. The case for n even is the same.

(d) Now $H^q(\mathbb{R}P^n) = H^q(S^n)^I \subset H^q(S^n)$, which proves that $H^q(\mathbb{R}P^n) = 0$ for $0 < q < n$.

For $q = 0$, we know that the cohomology is $\mathbb{R}$ since we have a nontrivial invariant form on S^n, namely $(f + i^*f)/2$.

For $q = n$, the previous part shows that there is a nontrivial invariant cohomology class if and only if n is odd, which concludes the proof. □

Exercise 6.50 (p.79)**.** If $f, g : S \to M$ are homotopic maps, show that $H^*(f)$ and $H^*(g)$ are isomorphic algebras.

Solution. This is a pretty standard fact, and is just an application of the five lemma using the homotopy between f and g. □

§7 The Nonorientable Case

Exercise 7.9 (p.87)**.** Let M be a manifold of dimension n. Computer the cohomology groups $H_c^n(M)$, $H^n(M, L)$, and $H_c^n(M, L)$ for each of the following four cases: M compact orientable, noncompact orientable, compact nonorientable, noncompact nonorientable.

Solution. We claim that we have the following table, where we incorporate the information from Corollary 7.8.1.

	$H^n(M)$	$H_c^n(M)$	$H^n(M, L)$	$H_c^n(M, L)$
compact orientable	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
noncompact orientable	0	$\mathbb{R}$	0	$\mathbb{R}$
compact nonorientable	0	0	$\mathbb{R}$	$\mathbb{R}$
noncompact nonorientable	0	0	0	$\mathbb{R}$

For orientable M, we know that $H^n(M) = H^n(M, L)$, and similarly for the compactly supported versions. This gives the first two rows of the table.

On the other hand, for nonorientable M, the same argument as in Corollary 7.8.1 shows that $H_c^n(M) = H^0(M, L) = 0$. Finally, we know that $H^n(M, L) = H_c^0(M)$ and $H_c^n(M, L) = H^0(M)$. Now $H_c^0(M)$ is spanned by the constant functions, i.e., is $\mathbb{R}$, regardless of the compactness of M. Similarly $H^0(M)$ is spanned by the *compactly supported* constant functions; this is $\mathbb{R}$ if M is compact and 0 otherwise. This gives the final two rows. □

Exercise 7.11 (p.88)**.** Compute the twisted de Rham cohomology $H^*(\mathbb{R}P^n, L)$.

Solution. We know that $H^*(\mathbb{R}P^n, L) = H_c^{n-*}(\mathbb{R}P^n) = H^{n-*}(\mathbb{R}P^n)$. Thus we conclude that

$$H^*(\mathbb{R}P^n, L) = \begin{cases} \mathbb{R} & \text{for } q = 0, \\ 0 & \text{for } 0 < q < n, \\ \mathbb{R} & \text{for } q = n \text{ odd}, \\ 0 & \text{for } q = n \text{ even} \end{cases}$$

by Exercise 6.46, which computed the de Rham cohomology of $\mathbb{R}P^n$. □

Chapter 2

The Čech–de Rham Complex

§8 The Generalized Mayer–Vietoris Principle

> **Exercise 8.4** (p.93). Suppose $\alpha < \beta$. Then $(\delta\omega)_{\ldots\beta\ldots\alpha\ldots}$ may be defined either as $-(\delta\omega)_{\ldots\alpha\ldots\beta\ldots}$ or by the difference operator formula (8.2). Show that these two definitions agree.

Solution. Say $k < j$ with $\beta = \alpha_k > \alpha_j = \alpha$. Then the difference operator formula tells us that

$$(\delta\omega)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=0}^{p+1}(-1)^i\omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+1}},$$

while the $-(\delta\omega)_{\ldots\alpha\ldots\beta\ldots}$ formula gives us

$$-(\delta\omega)_{\alpha_0\ldots\alpha\ldots\beta\ldots\alpha_{p+1}} = \sum_{i=0}^{k}(-1)^{i+1}\omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha\ldots\beta\ldots\alpha_{p+1}} + \sum_{i=k+1}^{j}(-1)^{i+1}\omega_{\alpha_0\ldots\alpha\ldots\hat{\alpha}_i\ldots\beta\ldots\alpha_{p+1}} + \sum_{i=j+1}^{p+1}(-1)^{i+1}\omega_{\alpha_0\ldots\alpha\ldots\beta\ldots\hat{\alpha}_i\ldots\alpha_{p+1}}.$$

Splitting up the first formula to be more similar to the $-(\delta\omega)$ formula gives us

$$(\delta\omega)_{\alpha_0\ldots\beta\ldots\alpha\ldots\alpha_{p+1}} = \sum_{i=0}^{k}(-1)^{i}\omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\beta\ldots\alpha\ldots\alpha_{p+1}} + \sum_{i=k+1}^{j}(-1)^{i}\omega_{\alpha_0\ldots\beta\ldots\hat{\alpha}_i\ldots\alpha\ldots\alpha_{p+1}} + \sum_{i=j+1}^{p+1}(-1)^{i}\omega_{\alpha_0\ldots\beta\ldots\alpha\ldots\hat{\alpha}_i\ldots\alpha_{p+1}}.$$

Induction on p and matching like terms gives us the result. Note that the base case $p = 2$ is fine, since if $\alpha < \beta$ then

$$-(\delta\omega)_{\alpha\beta} = \omega_\alpha - \omega_\beta = (\delta\omega)_{\beta\alpha}.$$

□

§9 More Examples and Applications of the Mayer–Vietoris Principle

> **Exercise 9.7** (p.104). Show that $\delta f(\alpha) = 0$. (Here f is the collating formula from Proposition 9.5.)

Solution. Expanding out, we want to show that

$$\delta\left[\sum_{i=0}^{n}(-D''K)^i\alpha_i - \sum_{i=1}^{n+1}K(-D''K)^{i-1}\beta_i\right] = 0. \tag{9.1}$$

By taking out the $i = 0$ term, we see that the first term becomes

$$\delta\sum_{i=0}^{n}(-D''K)^i\alpha_i = \sum_{i=1}^{n}(-1)^i(-D''K)^i\alpha_i + \delta\alpha_0.$$

On the other hand, the second term becomes

$$\partial K \sum_{i=1}^{n+1} (-1)^{i-1} (D''K)^{i-1} (\partial \alpha_{i-1} + D'' \alpha_i),$$

which is in turn equal to

$$\partial K \left[\partial \alpha_0 + \sum_{i=2}^{n+1} (-1)^{i-1} (D''K)^{i-1} \partial \alpha_{i-1} + \sum_{i=1}^{n} (-1)^{i-1} (D''K)^{i-1} D'' \alpha_i + (-1)^n (D''K)^n D'' \alpha_{n+1} \right].$$

The two middle terms are just

$$\partial K \left[\sum_{i=1}^{n} (-1)^i \left((D''K)^i \partial - (D''K)^{i-1} D'' \right) (\alpha_i) \right] = \partial K \left[\sum_{i=1}^{n} (-1)^i \partial (D''K)^i (\alpha_i) \right]$$

by Lemma 9.6. Thus the second term in Equation (9.1) is just

$$\partial K \partial \alpha_0 + \partial K \left(\sum_{i=1}^{n-1} (-1)^i \partial (D''K)^i (\alpha_i) \right) + \partial K (-1)^{n-1} (D''K)^{n-1} D'' \alpha_n.$$

Notice, furthermore, that

$$\partial \alpha_0 - \partial K \partial \alpha_0 = \partial [\alpha_0 - K \partial \alpha_0] = \partial^2 K \alpha_0 = 0.$$

Since $\alpha_{n+1} = 0$, it is therefore sufficient to show that

$$\sum_{i=1}^{n} (-1)^i \left[\partial (D''K)^i - \partial K \partial (D''K)^i \right] (\alpha_i) = 0.$$

But we know that $\partial K + K\partial = 1$, and so

$$\partial (D''K)^i - \partial K \partial (D''K)^i = (1 - \partial K)(\partial (D''K)^i) = K \partial^2 (D''K)^i = 0,$$

since $\partial^2 = 0$. This proves the result. □

> **Exercise 9.10** (p.105)**.** The real projective plane $\mathbb{R}P^2$ is obtained by identifying the boundary of a disk (see Figure 9.5 in the book). Find a good cover for $\mathbb{R}P^2$ and compute its de Rham cohomology from the combinatorics of the cover.

Solution. We use the hint in Figure 9.6 of the textbook, which gives the nerve of one possible good cover. An example of a good cover with that nerve is shown in Figure 3 below. Number the green, blue, and red domains on the left side as 0, 1, 2 and

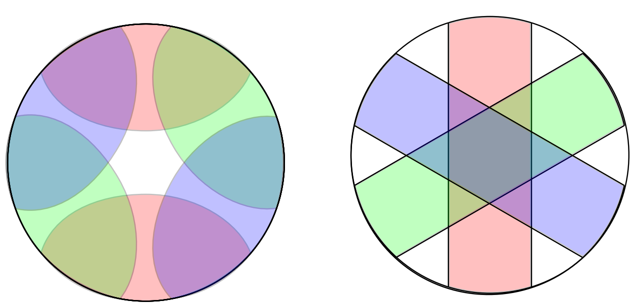


Figure 3: A good cover of $\mathbb{R}P^2$ made of the six shaded domains above. (I messed up the diagram slightly; the strips should be rotated 30° clockwise so that, for example, the red strip only intersects the red and green circles.)

on the right side as 3, 4, 5, respectively. Further, let $\mathfrak{U}$ be the cover. Then

$$C^0(\mathfrak{U}, \mathbb{R}) = \mathbb{R}^6, \quad C^1(\mathfrak{U}, \mathbb{R}) = \mathbb{R}^{15}, C^2(\mathfrak{U}, \mathbb{R}) = \mathbb{R}^{10}.$$

As in the examples, we see that $\ker \delta_0$ consists of $(\omega_0, \dots, \omega_5)$ such that $\omega_0 = \cdots = \omega_5$, and hence is 1-dimensional. In particular, we have $\ker \delta_0 = \mathbb{R}$ and $\operatorname{im} \delta_0 = \mathbb{R}^5$.

Now $\omega \in \ker \delta_1$ implies that $\omega_{13} - \omega_{03} + \omega_{01} = 0$, and so on. In particular, the values of ω along two edges of a triangle in the nerve (Figure 9.6) determine the value of ω along the third edge. We can check, then, that $\omega_{01}, \omega_{13}, \omega_{02}, \omega_{35}, \omega_{34}, \omega_{12}$ is enough to determine ω everywhere. Furthermore, there is a linear equation which these must satisfy in order to be a valid (i.e., consistently defined) ω.

To put it perhaps overly explicitly: Determining ω_{01} and ω_{13} gives ω_{03}. Then ω_{02} give ω_{23}. 35 gives 15, and 34 gives 24 and 25. Finally 12 gives 14 and 25, which then give 05 and 14 as part of the triangles 025 and 014. This seemingly gives $\ker \delta_1 = \mathbb{R}^6$, but 05 and 14 must additionally satisfy a triangle equation with 045.

Hence $\ker \delta_1$ actually has five degrees of freedom. That is to say, $\ker \delta_1 = \mathbb{R}^5$ and $\operatorname{im} \delta_1 = \mathbb{R}^{10}$.

Thus we have

$$H_0 = \mathbb{R}, \quad H_1 = \mathbb{R}^5/\mathbb{R}^5 = 0, \quad H_2 = \mathbb{R}^{10}/\mathbb{R}^{10} = 0,$$

which is what we would expect from de Rham cohomology. □

> **Exercise 9.11** (p.105)**.** Figure 9.7 (in the book) shows the nerve of a good cover $\mathfrak{U}$ on the torus, where the arrows indicate how the vertices are ordered. Write down a nontrivial 1-cocycle in $C^1(\mathfrak{U}, \mathbb{R})$.

Solution. We are looking for some $\eta = (\eta_{01}, \eta_{02}, \dots)$ which satisfies the alternating sum relation on each triangle in the figure, but where $\eta_{\alpha\beta}$ cannot be consistently written as $\omega_\beta - \omega_\alpha$.

Consider assigning 1 to every horizontal and vertical edge, and 2 to every diagonal edge. This gives an element η which is clearly a 1-cocycle. Suppose $\eta = \delta\omega$. Label the vertices in ascending order from left to right, then bottom to top (i.e., 0 on the bottom left and 20 on the top right). Now observe that

$$0 = \eta_{02} = \omega_2 - \omega_0 = (\omega_2 - \omega_1) + (\omega_1 - \omega_0) = 2.$$

This is of course not true, and so η is a nontrivial 1-cocycle, as desired. □

> **Exercise 9.13** (p.108)**.** Give a proof of Step 2 of the proof of Proposition 9.12 (the Künneth formula). In particular, this is the following statement: Whenever a homomorphism $f : K \to L$ of double complexes induces H_d-isomorphism, it also induces H_D-isomorphism.

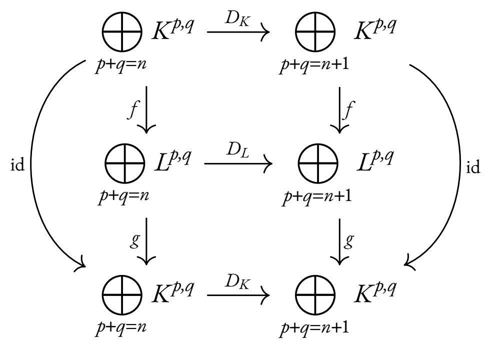
Solution. Recall that D is the total differential $D : \bigoplus_{p+q=n} K^{p,q} \to \bigoplus_{p+q=n+1} K^{p,q}$. We know that

$$\begin{array}{ccc} \bigoplus_{p+q=n} K^{p,q} & \longrightarrow & \bigoplus_{p+q=n+1} K^{p,q} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \bigoplus_{p+q=n} L^{p,q} & \longrightarrow & \bigoplus_{p+q=n+1} L^{p,q} \end{array}$$

commutes. In particular, since $D = \delta + (-1)^p d$ and f commutes with both D and d, we know that the diagram commutes when restricted to any given $K^{p,q}$, and hence commutes on $\bigoplus_{p+q=n} K^{p,q}$ as well.

Let $g : L \to K$ be the inverse H_d-isomorphism. It suffices to prove that g is an H_D-homomorphism, since we already know that it is bijective.

Consider the following diagram:



We know that $D_K gf = gfD_K = gD_L f$. Thus we need only be able to cancel f. But f is an isomorphism on each term. □

§10 Presheaves and Čech Cohomology

> **Exercise 10.5** (p.111)**.** Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover and $\mathfrak{B} = \{V_\beta\}_{\beta\in J}$ be a refinement. Let $\phi, \psi : J \to I$ be two refinement maps. Show that $\psi^\# - \phi^\# = \partial K + K\partial$.

Solution. Let $\omega \in C^q(\mathfrak{U}, \mathscr{F})$. Then

$$((\psi^\# - \phi^\#)\omega)\left(V_{\beta_0\dots\beta_{q-1}}\right) = \omega\left(U_{\psi(\beta_0)\dots\psi(\beta_{q-1})}\right) - \omega\left(U_{\phi(\beta_0)\dots\phi(\beta_{q-1})}\right).$$

On the other hand, we know that

$$\begin{aligned}(\partial K\omega)\left(V_{\beta_0\dots\beta_{q-1}}\right) &= \partial\sum(-1)^i\omega\left(U_{\phi(\beta_0)\dots\phi(\beta_i)\psi(\beta_i)\dots\psi(\beta_{q-1})}\right)\\ &= \sum_{i=0}^{q-1}\sum_{j=0}^{q}(-1)^{i+j}\omega\left(U_{\phi(\beta_0)\dots}\right),\end{aligned}$$

where the final subscript is obtained by deleting the j-th term in $\phi(\beta_0)\dots\phi(\beta_i)\psi(\beta_i)\dots\psi(\beta_{q-1})$. Breaking the sum up, we see that this is equal to

$$\sum_{i=0}^{q-1}\left[\sum_{j=0}^{i}(-1)^{i+j}\omega\left(U_{\phi(\beta_0)\dots\widehat{\phi(\beta_j)}\dots\phi(\beta_i)\psi(\beta_i)\dots\psi(\beta_{q-1})}\right) + \sum_{j=i}^{q-1}(-1)^{i+j-1}\omega\left(U_{\phi(\beta_0)\dots\phi(\beta_i)\psi(\beta_i)\dots\widehat{\psi(\beta_j)}\dots\psi(\beta_{q-1})}\right)\right]. \tag{10.1}$$

Furthermore, we know that

$$K\partial\omega\left(V_{\beta_0\dots\beta_{q-1}}\right) = K\sum_{i=0}^{q-1}(-1)^i\omega\left(V_{\beta_0\dots\widehat{\beta_i}\dots\beta_{q-1}}\right).$$

This is just

$$\sum_{i=0}^{q-1}(-1)^i\sum_{j=0}^{q-2}\omega\left(U_{\phi(\beta_0)\dots}\right).$$

The final subscript is obtained by removing all β_i terms, and then repeating ϕ and ψ up to the j-th β remaining, which is β_j if $j < i$ and is β_{j+1} if $j \geq i$. In particular, writing this out and swapping i and j to be more in line with the ∂K expression, this is equal to

$$\sum_{j=0}^{q-1}\left[\sum_{i=0}^{j-1}(-1)^{i+j}\omega\left(U_{\phi(\beta_0)\dots\phi(\beta_i)\psi(\beta_i)\dots\widehat{\psi(\beta_j)}\dots\psi(\beta_{q-1})}\right) + \sum_{i=j+1}^{q-1}(-1)^{i+j-1}\omega\left(U_{\phi(\beta_0)\dots\widehat{\phi(\beta_j)}\dots\phi(\beta_i)\psi(\beta_i)\dots\psi(\beta_{q-1})}\right)\right]. \tag{10.2}$$

Now the term $U_{\dots}$ where the subscript copies β_i and deletes β_j for $i < j$ is assigned a sign $(-1)^{i+j-1}$ in Equation (10.1) and a sign $(-1)^{i+j}$ in Equation (10.2). Thus these terms cancel. Similarly the terms for $i > j$ cancel. The only leftover terms are from Equation (10.1) where $i = j$. If $i = j$ is neither 0 nor $q - 1$, then this copy of U appears in both terms in Equation (10.1) and indeed cancels. Thus we are left with

$$((\partial K + K\partial)\omega)\left(V_{\beta_0\dots\beta_{q-1}}\right) = \omega\left(U_{\psi(\beta_0\dots\psi(\beta_{q-1})}\right) + (-1)^{2(q-1)-1}\omega\left(U_{\phi(\beta_0)\dots\phi(\beta_{q-1})}\right),$$

which is exactly $\left((\psi^\# - \phi^\#)\omega\right)(V_{\dots})$. □

Exercise 10.7 (p.112)**.** Let $\mathscr{F}$ be the presheaf on S^1 which associates to every open set the group $\mathbb{Z}$. We define the restriction homomorphism on the good cover $\mathfrak{U} = \{U_0, U_1, U_2\}$ (Figure 10.1 in the book) by

$$\begin{aligned}\rho_{01}^0 &= \rho_{01}^1 = 1,\\ \rho_{12}^1 &= \rho_{12}^2 = 1,\\ \rho_{02}^2 &= -1, \rho_{02}^0 = 1,\end{aligned}$$

where ρ_{ij}^i is the restriction from U_i to $U_i \cap U_j$. Computer $H^*(\mathfrak{U}, \mathscr{F})$.

Solution. The chain complex looks like

$$0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\delta} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

Here we use the order U_0, U_1, U_2 for C^0 and U_{12}, U_{02}, U_{01} for C^1.

Consider the map $\partial_0 : C^1 \to C^0$. Then $\mathscr{F}(\partial_0) : C^0 \to C^1$ takes $(a, b, c) \in \mathscr{F}(U_0) \oplus \mathscr{F}(U_1) \oplus \mathscr{F}(U_2)$ to

$$\left(\rho_{12}^2(a, b, c), \rho_{02}^2(a, b, c), \rho_{01}^1(a, b, c)\right) = (c, -c, a).$$

Similarly $\mathscr{F}(\partial_1)$ takes $(a, b, c) \mapsto (b, a, a)$. Hence $\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})$ takes $(a, b, c) \in \mathbb{Z}^3$ to $(c-b, -c-a, b-a) \in \mathbb{Z}^3$. This homomorphism has kernel 0 and image $\mathbb{Z}\oplus\mathbb{Z}\oplus 2\mathbb{Z}$. Thus $H^0(\mathfrak{U}, \mathscr{F}) = 0$ while $H^1(\mathfrak{U}, \mathscr{F}) = \mathbb{Z}/2$. □

§11 Sphere Bundles

Exercise 11.10 (p.122)**.** If $s : M \to E$ is a section, show that $Ks^* = s^*K$.

Solution. Notice that

$$(s^*K\omega)_{\alpha_0\dots\alpha_{p-1}} = s^*\left(\sum \pi^*\rho_\alpha\omega_{\alpha\alpha_0\dots\alpha_{p-1}}\right) = \sum s^*\pi^*\rho_\alpha\omega_{\alpha\alpha_0\dots\alpha_{p-1}}.$$

But since $s^*\pi^* = (\pi \circ s)^* = \mathrm{id}$, we know that this is exactly equal to

$$\sum \rho_\alpha\omega_{\alpha\alpha_0\dots\alpha_{p-1}},$$

which is by definition $(Ks^*\omega)_{\alpha_0\dots\alpha_{p-1}}$. □

Exercise 11.13 (p.122)**.** Use the existence of the global angular form ψ to prove Proposition 11.9, i.e., that if the oriented sphere bundle E has a section, then its Euler class vanishes.

Solution. Let $s : M \to E$ be a section of the oriented sphere bundle. The global angular form ψ such that $d\psi = -\pi^*e$, so that $e = -s^*d\psi = -ds^*\psi$ is 0 in cohomology. □

> **Exercise 11.19** (p.126)**.** Show that the Euler class of an oriented sphere bundle E with even-dimensional fibers is zero, at least when the sphere bundle comes from a vector bundle.

Solution. Recall that S^{2n} comes with an orientation-reversing differeomorphism, namely the antipodal map $a : x \mapsto -x$. Call A the map on E obtained by applying a to each fiber. Let E' be the image of this map. Note that E' has is bundle isomorphic to E, but with the opposite orientation. Hence $e(E) = -e(E')$.

We claim, however, that $e(E) = e(E')$ as well. To see this, let $\{U_\alpha\}$ be a good cover of M and let $\{\sigma_\alpha\}$ be generators of $H^n(E|_{U_\alpha})$ with $[\sigma_\alpha] = [\sigma_\beta]$ on $U_{\alpha\beta}$. (Note that the σ_α's define an orientation on E, not on E', which has the reverse orientation.) Recall that $\{\sigma_\alpha\} \in \sigma^{0,n}$ in the tic-tac-toe diagram and give Euler class $e(E)$ represented by $-\pi^*\varepsilon$.

Furthermore, there is a double complex isomorphism A^* obtained by taking $\prod \omega_{\alpha_0\ldots\alpha_p}$ to $\prod A^*\omega_{\alpha_0\ldots\alpha_p}$. (In the final expression, we suppress the restriction operation; technically A^* should be restricted to $U_{\alpha_0\ldots\alpha_p}$.) This is a double complex homomorphism because pullback commutes with d and is linear, hence commutes with the difference operator δ. Furthermore, it is an isomorphism because $A^2 = \text{id}$.

Now applying A^* to the tic-tac-toe diagram and thus getting a tic-tac-toe diagram for E' instead, we see that $e(E')$ is represented by $A^*(-\pi^*\varepsilon) = -(\pi \circ A)^*\varepsilon$. But $\pi \circ A = \pi$, and so $e(E')$ is represented by $-\pi^*\varepsilon$ too, as desired. □

> **Exercise 11.21** (p.126)**.** Compute the Euler class of the unit tangent bundle of the sphere S^k by finding a vector field on S^k and computing its local degrees.

Solution. Embed S^k in $\mathbb{R}^{k+1}$ and consider the vector field of S^k given by

$$W_p = W_{(p_0,\ldots,p_k)} = \left(\frac{p_0 p_k}{p_k+1}, \ldots, \frac{p_{k-1}p_k}{p_k+1}, -(1-p_k)\right).$$

Then clearly V_p is tangent to S^k at p since

$$W_p \cdot \vec{p} = \frac{p_k}{p_k+1}(1-p_k^2) - p_k(1-p_k) = p_k(1-p_k) + p_k(1-p_k) = 0.$$

Then let V_p be the unit vector in the direction of W_p, namely $V_p = \frac{W_p}{\|W_p\|}$. To write this even more explicitly, note that

$$\|W_p\|^2 = \frac{p_k^2(1-p_k^2)}{(1+p_k)^2} + (1-p_k)^2 = \frac{1-p_k^2}{(1+p_k)^2} = \frac{1-p_k}{1+p_k}.$$

Note that V_p is defined everywhere on S^k except when $p_k = -1$, i.e., at the south pole $S = (0, \ldots, 0, -1)$, and when $\|W_p\| = 0$. This last condition occurs exactly when $p_k = 1$, i.e., at the north pole $N = (0, \ldots, 0, 1)$.

Hence it is sufficient to calculate the local degrees at N and S. We will do the calculation for S.

Let D_ε be the disk at S consisting of all points of S^k whose final coordinate is $-1+\varepsilon$. Then the local degree at S is given by the degree of the map $f : \partial D_\varepsilon \to S^{k-1}$ which takes $p = (p_0, \ldots, p_{k-1}, -1+\varepsilon) \in D_\varepsilon$ to $V_p \in S^{k-1}$.

> *Note.* A priori we only know that $V_p \in S^k$. But, if $p \in \partial D_\varepsilon$, then we know that $p_k = -1+\varepsilon$. Hence the final coordinate of V_p for $p \in \partial D_\varepsilon$ is
>
> $$-\frac{(1-p_k)}{\sqrt{(1-p_k)/(1+p_k)}} = -\sqrt{1-p_k^2} = -\sqrt{2\varepsilon-\varepsilon^2}$$
>
> on D_ε. This is of course fixed, so we can consider $V_p \in S^{k-1}$ in fact. Note that the first coordinates of V_p give a vector in $\mathbb{R}^k$ which is distance $\sqrt{1+\varepsilon^2-2\varepsilon} = 1-\varepsilon = |p_k|$ from the origin. Thus, when we consider $V_p \in S^{k-1}$, we actually scale these coordinates by $1/(1-\varepsilon)$.

Let σ be the standard volume form of S^{k-1}, namely

$$\sigma = \sum_{j=0}^{k-1}(-1)^j x_j dx_0 \ldots \widehat{dx_j} \ldots dx_{k-1}.$$

Then we know that

$$\int_{\partial D_\varepsilon} f^*\sigma = \int_{\partial D_\varepsilon} \sum_{j=0}^{k-1} (-1)^j (x_j \circ f) d(x_0 \circ f) \dots \widehat{d(x_j \circ f)} \dots d(x_{k-1} \circ f).$$

But, for $j = 0, \dots, k-1$, notice that $f_j = x_j \circ f$ is simply

$$\frac{p_j p_k}{p_k + 1} \cdot \sqrt{\frac{1 + p_k}{1 - p_k}} \cdot \frac{1}{\varepsilon - 1} = \frac{p_j p_k}{\sqrt{1 - p_k^2}} \cdot \frac{1}{p_k} = \frac{p_j}{\sqrt{1 - p_k^2}}.$$

Hence it follows that

$$\int_{\partial D_\varepsilon} f^*\sigma = \int_{\partial D_\varepsilon} \sum_{j=0}^{k-1} (-1)^j \frac{p_j}{(\sqrt{1 - p_k^2})^k} dp_0 \dots \widehat{dp_j} \dots dp_{k-1}.$$

Now we must calculate the volume form ω on D_ε. But notice that D_ε embeds into S^k via

$$i(a_0, \dots, a_{k-1}) = (a_0 \sqrt{1 - p_k^2}, \dots, a_{k-1} \sqrt{1 - p_k^2}, p_k),$$

where $p_k = -1 + \varepsilon$. It follows that the volume form, using coordinates $(x_0, \dots, x_k)$ for $S^k \in \mathbb{R}^{k+1}$, is just

$$\omega = \sum_j (-1)^j a_j da_0 \dots \widehat{da_j} \dots da_{k-1} = \sum_j (-1)^j \left(\frac{1}{\sqrt{1 - p_k^2}} \right)^k x_j \, dx_0 \dots \widehat{dx_j} \dots dx_{k-1}.$$

Hence the integrand of $f^*\sigma$ around a disk centered at the south pole S is exactly the generator of the top cohomology. The local degree at S is therefore 1.

Similarly, we can calculate that the local degree at N is 1. Thus the Euler number is 2, and the Euler class is twice the generator. □

Exercise 11.26 (p.129)**.** Let $f : M \to M$ be a smooth map of a compact oriented manifold to itself. Denote by $H^q(f)$ the induced map on the cohomology $H^q(M)$. The *Lefschetz number* of f is defined to be

$$L(f) = \sum_q (-1)^q \operatorname{trace} H^q(f).$$

Let Γ be the graph of f in $M \times M$.

(a) Show that

$$\int_\Delta \eta_\Gamma = L(f).$$

(b) Show that if f has no fixed points, then $L(f)$ is zero.

(c) At a fixed point P of f, the derivative $(Df)_P$ is an endomorhpism of the tangent space $T_P M$. We define the *multiplicity* of the fixed point P to be

$$\sigma_P = \operatorname{sgn} \det((Df)_P - I).$$

Show that if the graph Γ is transversal to the diagonal Δ in $M \times M$, then

$$L(f) = \sum_P \sigma_P,$$

where P ranges over the fixed points of f.

Solution.

(a) First note that $\eta_\triangle$ satisfies
$$\int_\triangle i^*\omega = \int_{M\times M} \omega \wedge \eta_\triangle$$
for any ω in $H^n(M)$. Here $n = \dim M$ and $i : \triangle \to M \times M$ is the inclusion. Since $\eta_\Gamma \in H^n(M)$, we know that
$$\int_\triangle \eta_\Gamma = \int_{M\times M} \eta_\Gamma \wedge \eta_\triangle = (-1)^{(\deg \eta_\Gamma)(\deg \eta_\triangle)} \int_{M\times M} \eta_\triangle \wedge \eta_\Gamma.$$
Using the explicit formula for $\eta_\triangle$ and letting $j : \Gamma \to M \times M$ be the inclusion, we see that the integral of $\eta_\triangle \wedge \eta_\Gamma$ is
$$\int_\Gamma j^*\eta_\triangle = \int_\Gamma j^*\left[\sum (-1)^{\deg \omega_i} \pi^*\omega_i \wedge \rho^*\tau_i\right] = \sum_i \int_\Gamma (-1)^{\deg \omega_i} j^*\pi^*\omega_i \wedge j^*\rho^*\tau_i.$$
Note that $j^*\pi^* = \mathrm{id}$ while $j^*\rho^* = f^*$. Furthermore, since $\deg \eta_\Gamma = \deg \eta_\triangle = n$, we have that
$$\int_\triangle \eta_\Gamma = (-1)^n \sum_i \int_\Gamma (-1)^{\deg \omega_i} \omega_i \wedge f^*\tau_i.$$
Each $f^*\tau_i$ is a linear combination $\sum c_{ij}\tau_j$. Then
$$\int_\Gamma \omega_i \wedge f^*\tau_i = \int_\Gamma \sum_j \omega_i \wedge c_{ij}\tau_j = c_{ii}$$
because the ω's and τ's are dual bases. In fact, because $n = \deg \omega_i + \deg \tau_i$, we have
$$\int_\triangle \eta_\Gamma = (-1)^{n^2} \sum_i (-1)^{\deg \omega_i} c_{ii} = (-1)^{n^2+n} \sum_i (-1)^{\deg \tau_i} c_{ii} = \sum_i (-1)^{\deg \tau_i} c_{ii}.$$
Now recall that c_{ii} is the coefficient of τ_i in $f^*\tau_i$. Thus if we take all the τ_i's of degree q, then we get $(-1)^q$ trace $H^q(f)$. Summing over all q, i.e., over all τ_i, gets us the desired equation.

(b) Recall (Equation 6.31) that
$$L(f) = \int_\triangle \eta_\Gamma = \int_{M\times M} \eta_\Gamma \wedge \eta_\triangle = \int_{M\times M} \eta_{\Gamma\cap\triangle},$$
where we suppress all the pullbacks by inclusions. But Γ and $\triangle$ don't intersect, so this is 0.

(c) Using the explicit formula for the Poincaré dual, we have
$$\int_\triangle \eta_\Gamma = \int_\triangle j_*\Phi_{N\Gamma}.$$
Since Γ and $\triangle$ are transverse, we know that $N\Gamma = T\triangle$. Thus integration is just integration along the fiber. Furthermore, we know that the integral $\int_\triangle \eta_\Gamma = \int_{M\times M} \eta_{\Gamma\cap\triangle}$, and $\eta_{\Gamma\cap\triangle}$ is 0 outside of the intersection points $\Gamma \cap \triangle$. Thus it is sufficient to find the integral in arbitrarily small neighborhoods around the intersection points, i.e., to find the integral along the fiber over $p \in \Gamma \cap \triangle$. Whether this integral is ± 1 depends on its multiplicity, proving the result. □

§12 Thom Isomorphism and Poincaré Duality Revisited

Exercise 12.6 (p.133)**.** Let $\pi : E \to M$ be an oriented vector bundle.

(a) Show that $\pi^*e = \Phi$ as cohomology classes in $H^*(E)$, but not in $H^*_{cv}(E)$.

(b) Prove that $\Phi \wedge \Phi = \Phi \wedge \pi^*e$ in $H^*_{cv}(E)$.

Solution.

(a)

(b) □

Exercise 12.9 (p.137)**.** Show that h and g in the proof of Proposition 12.1 are well-defined.

Exercise 12.10 (p.138)**.** Let $\mathbb{C}P^n$ have homogeneous coordinates $z_0, \dots, z_n$. Define $U_i = \{z_i \neq 0\}$, so that $\mathfrak{U} = \{U_0, \dots, U_n\}$ is an open cover of $\mathbb{C}P^n$, although not a good cover. Computer $H^*(\mathbb{C}P^n)$ from the double complex $C^*(\mathfrak{U}, \Omega^*)$. Find elements in $C^*(\mathfrak{U}, \Omega^*)$ which represent the generators of $H^*(\mathbb{C}P^n)$.

Exercise 12.11 (p.138)**.** Apply the Thom isomorphism (12.2) to compute the cohomology with compact support of the open Möbius strip (cf. Exercise 4.8).

Exercise 12.12.1 (p.140)**.** Show that the definition of τ in the proof above provides a homotopy operator for the compact Mayer–Vietoris sequence (12.12). More precisely, if ω is in $\bigoplus \Omega_c^*(U_{\alpha_0 \dots \alpha_p})$ and

$$(K\omega)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \rho_{\alpha_i} \omega_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}},$$

then

$$\partial K + K\partial = 1.$$

Exercise 12.16 (p.141)**.** By applying Poincaré duality (12.15), compute the compact cohomology of the open Möbius strip (cf. Exercise 4.8).

§13 Monodromy

Exercise 13.6 (p.152)**.** Since H_d of the double complex $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$ in Example 13.5 has only one nonzero row, we see by the generalized Mayer–Vietoris principle and Proposition 12.1 that

$$H^*(S^1) = H_D^*\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\} = H_\delta H_d = H^*(\mathfrak{U}, \mathcal{H}^0).$$

Compute the Čech cohomology $H^*(\mathfrak{U}, \mathcal{H}^0)$ directly.

Exercise 13.8 (p.152)**.** As in Example 13.5, with $\mathfrak{U}$ being the usual good cover of S^1,

$$H^*(\mathbb{R}^1) = H_D^*\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\} = H_\delta H_d = H^*(\mathfrak{U}, \mathcal{H}^0).$$

Compute $H^*(\mathfrak{U}, \mathcal{H}^0)$ directly.

Exercise 13.10 (p.153)**.** Let $X = S^1 \wedge S^2$. Find the homotopy type of the space E defined by

$$E = \widetilde{X} \times I/(x, 0) \sim (s(x), 1),$$

where s is the deck transformation of the universal cover $\widetilde{X}$ which shifts everything one unit up.

Chapter 3

Spectral Sequences and Applications

Chapter 4

Characteristic Classes