An introduction to Khovanov homology

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Abstract. In 2000, Mikhail Khovanov introduced a categorification of the famous Jones polynomial. This categorification, now known as Khovanov homology, is an invariant of knots and links. Its construction involves building a so-called "cube of resolutions" of a diagram of a knot, and then applying a certain topological quantum field theory (TQFT) to the cube; different choices of TQFT give rise to different variants of Khovanov homology. In this paper, we describe the construction of the chain complex, verify invariance of the resulting homology, and apply two different TQFTs to define two link homologies: Khovanov homology and a well-known variant called Lee homology.

1 The Khovanov chain complex

Let L be a link, and D some diagram of L. Number the crossings of D from 1 to k. Each crossing may be resolved, or smoothed, in one of two ways, as seen in Figure 1. If we resolve all of the crossings of D,

Figure 1: A crossing may be 0-smoothed or 1-smoothed. (Figure from [5].)

then we get a crossing-less diagram, i.e., a union of disjoint circles in the plane. We call such a diagram a *resolution* or *smoothing* of D. We label these smoothings by k-tuples of 0's and 1's. In particular, the tuple $v = (\varepsilon_1, \dots, \varepsilon_k) \in \{0,1\}^k$ corresponds to the smoothing where the i-th crossing is ε_i-smoothed. There are thus 2^k many smoothings of D. We denote the smoothing corresponding to v as D_v, and write $|D_v| = \sum \varepsilon_i$.

We consider these smoothings to be the "vertices" of the k-dimensional cube $[0,1]^k$. As with a usual cube, we draw an edge between two vertices if and only if they are identical in all but one coordinate. For example, we would connect the vertex $(0,\dots,0)$ with the k many vertices $e_1 = (1,0,\dots,0), \dots, e_k = (0,\dots,0,1)$. In other words, we connect the smoothing consisting of all 0-smoothings with the k many smoothings of D which have exactly one crossing which is given a 1-smoothing. If there is an edge between v and w, then $|v| = |w| \pm 1$. We direct the edge to go from the vertex with smaller norm to the vertex with larger norm. In particular, if there is an edge from v to w, then $|v| = |w| - 1$. We write the head v of an edge as $h(e)$, and its tail w as $t(e)$.

Each edge e goes from one smoothing to another, that is to say, from one union of circles to another. Furthermore, the only difference between the two vertices of e is that one of them has some crossing 0-resolved, and the other has it 1-resolved. Thus the two vertices differ only in the neighborhood of a crossing.

If e goes from v to w, then consider the cobordism which is the identity cobordism $(D_v - N) \times I = (D_w - N) \times I$ away from this neighborhood N, and which looks like the left side of Figure 2 in this neighborhood. We call this cobordism $S_e : D_v \to D_w$; it looks like one of the diagrams on the right side of

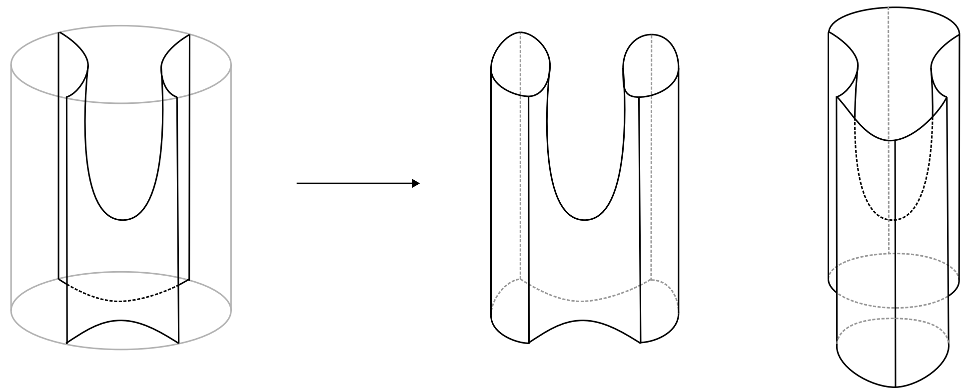

Figure 2: The standard cobordism from a 0-smoothing to a 1-smoothing is depicted on the left. (Note that we draw our cobordisms from bottom to top.) When combined with the product cobordism on the rest of the smoothing, we get one of the two diagrams on the right. In the first case, we call the cobordism a "split"; in the second, we call it a "merge."

Figure 2, along with a cylinder for each remaining circle in the resolution D_v.

Geometrically, we thus have a cube of resolutions whose vertices are unions of circles obtained as smoothings of D and whose edges are cobordisms between these smoothings.

The Khovanov complex is now obtained by applying a (1+1)-dimensional TQFT $\mathcal{F}$ to this cube. In particular, we consider the category $\mathcal{C}ob$ whose objects are closed 1-manifolds (i.e., unions of circles) and whose morphisms $\mathrm{Hom}(v, w)$ are 2-dimensional cobordisms whose boundary is $v \cup (-w)$. A TQFT is then a functor from $\mathcal{C}ob$ to the category of vector spaces. (In our case, our vector spaces are $\mathbb{Q}$-vector spaces.) More concretely, we replace each vertex v in the cube by a vector space $\mathcal{F}(D_v)$, and an edge $e : v \to w$ by a vector space map $\mathcal{F}(S_e) : \mathcal{F}(D_v) \to \mathcal{F}(D_w)$.

It turns out that applying such a TQFT gives rise to a link homology theory. In particular, let the group $\mathrm{CKh}_{\mathcal{F}}(D)$ be the direct sum $\bigoplus \mathcal{F}(D_v)$ over all smoothings v. Where it is unambiguous, we write CKh instead of $\mathrm{CKh}_{\mathcal{F}}$. This group comes with a natural differential, defined as follows. First, if e is an edge from v to w, then v and w only differ at one crossing, say the i-th one. Then define $s(e)$ to be the number of crossings from 1 to i which are 1-smoothed in v (or, equivalently, in w). For example, if e goes from $(0, 1, 1, 0, 0)$ to $(0, 1, 1, 1, 0)$ then $i = 4$ and $s(e) = 2$. Now we may define the differential d to be the map taking $x \in \mathcal{F}(D_v)$ to

$$\sum_{\{e:h(e)=v\}} (-1)^{s(e)} (\mathcal{F}(S_e))(x) \in \bigoplus_{\{e:h(e)=v\}} \mathcal{F}(D_{t(e)}).$$

The differential always maps a vector x belonging to a smoothing of norm k to a linear combination of vectors belonging to smoothings of norm $k+1$. In particular, this gives rise to a homological grading on CKh. In keeping with convention, rather than defining the homological grading of $x \in \mathcal{F}(D_v)$ to be $|v|$, we define it to be $\mathrm{gr}(v) := |v| - n_-$, where n_- is the number of negative crossings in D.

At this point, we must prove that this definition does indeed give a chain complex.

Proposition 1. *We have $d^2(x) = 0$ for every element $x \in \mathrm{CKh}(D)$,*

Proof. Suppose $x \in \mathcal{F}(D_v)$. It suffices to show that, for every u which swaps exactly two 0-smoothings in v for 1-smoothings, the component of $d^2 x$ in $\mathcal{F}_u$ is zero. But this component is composed of two different pieces. In particular, suppose without loss of generality that $v = (0, 0, v')$ and $u = (1, 1, v')$. Then there is a piece of d^2x which factors through the smoothing $w_1 = (1, 0, v')$, and another piece which factors through $w_2 = (0, 1, v')$. We would like to show that these factors cancel each other out.

Notice that, ignoring signs, the cobordisms are the same. After all, geometrically, the cobordisms from v to w_1 and from w_2 to u are identical, as are the cobordisms from v to w_2 and from w_1 to u. Hence if we call the former A and the latter B, we only claim that $AB = BA$. This is true because the cobordisms are disjoint. Geometrically, this is described in Figure 3 below.

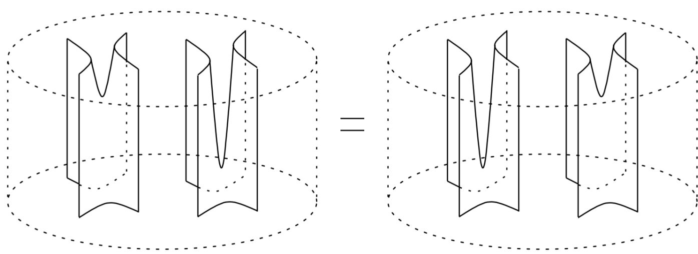

Figure 3: Up to sign, the two cobordisms from v to u are identical, and hence give the same map under the TQFT $\mathcal{F}$. (Figure from [2].)

On the other hand, the two cobordisms are given opposite signs: From v to w_2 to u, the sign is +1, as there is never a 1 to the left of the coordinate which is being changed. On the other hand, from v to w_1 is again a positive sign, while w_1 to u is a negative sign, since the second coordinate is changed but the first coordinate already is 1-smoothed. Hence the cobordism passing through w_1 is given a negative sign. This implies that d^2 is identically zero, as desired. □

Thus we may take homology to get a homology $\mathrm{Kh}_{\mathcal{F}}(D)$. (Technically, since d increases the homological grading, we are really taking *co*homology.) We claim that this is a link invariant, as long as we impose a few conditions on $\mathcal{F}$. In particular, we ask that $\mathcal{F}$ satisfy the so-called S, T, and $4Tu$ conditions.

To define these relations, first notice that a closed surface can be thought of as a cobordism from $\emptyset$ to $\emptyset$. Furthermore, the TQFT takes the empty set to the base ring; in our case, we typically have $\mathcal{F}(\emptyset) = \mathbb{Q}$. Thus a closed surface may be identified with a map from $\mathbb{Q}$ to itself. Such a map is, in turn, determined by where it sends $1 \in \mathbb{Q}$, so that we may consider a closed surface to in fact be a rational number (or, more generally, an element in the base ring).

We ask that a sphere be assigned the number 0, while a torus (i.e., a genus-one surface) be assigned the number 2. This means that a cobordism with a closed sphere is always 0, and that we may replace a torus in a cobordism by multiplying the rest of the cobordism by a factor of 2. These are the S and T relations, respectively. The $4Tu$ (i.e., "four tubes") relation is somewhat more complicated. To begin, suppose C is a cobordism such that, in some ball, it looks like four disks $D_1, \ldots, D_4$. If C_{ij} is the result of removing the disks D_i and D_j, and replacing them with a tube with the same boundary, then we ask that $C_{12} + C_{34} = C_{13} + C_{24}$. See Figure 4.

Theorem 2. *For every (1+1)-dimensional TQFT $\mathcal{F}$, the homology $\mathrm{Kh}_{\mathcal{F}}(D)$ of $\mathrm{CKh}_{\mathcal{F}}(D)$ is a link invariant, which we may thus denote as $\mathrm{Kh}_{\mathcal{F}}(L)$. (Recall that D is a diagram for the link L.)*

Proof. It is enough to show that $\mathrm{Kh}_{\mathcal{F}}(D)$ is invariant under each of the three Reidemeister moves, which may be seen in Figure 5. For simplicity, we omit the subscript $\mathcal{F}$ for the remainder of this proof.

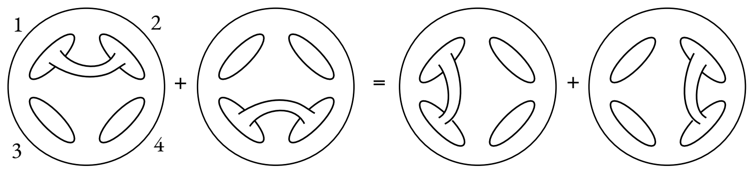

Figure 4: A local picture for the $4Tu$ relationship.

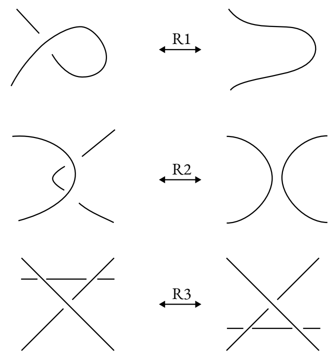

Figure 5: The three Reidemeister moves.

First, suppose D and D' are the two diagrams in the R1 move; in particular, let D be the diagram on the top left of Figure 5 and D' the one on the top right. Let c be the crossing which is in D but not D', i.e., the crossing shown in the figure. We want to show that the homology of $\mathrm{CKh}(D)$ and the homology of $\mathrm{CKh}(D')$ are isomorphic. We may do that by constructing chain maps $F : \mathrm{CKh}(D') \to \mathrm{CKh}(D)$ and $G : \mathrm{CKh}(D) \to \mathrm{CKh}(D')$ such that FG and FG are both homotopic to the identity. Then the chain complexes will be chain homotopy equivalent, hence will be isomorphic in homology.

First, notice that we may write $\mathrm{CKh}(D)$ as $\mathrm{CKh}(D_0) \oplus \mathrm{CKh}(D_1)$. Here D_0 is the diagram obtained by 0-smoothing the crossing c and D_1 is the diagram obtained by giving c the 1-resolution, as in Figure 6. Furthermore, let us assume without loss of generality that c is the first crossing. The grading of any element

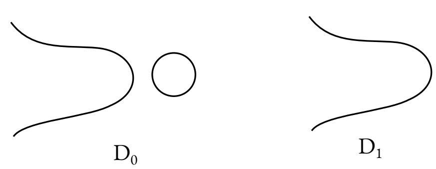

Figure 6: The two diagrams obtained by smoothing the crossing c in D.

$v \in \mathrm{CKh}(D_1)$ is $\mathrm{gr}_{D_1}(v) = |v| - n_-(D_1)$. On the other hand, seen as an element $(1, v) \in \mathrm{CKh}(D)$, it has grading $\mathrm{gr}_D(v) = |v| + 1 - n_-(D)$. Note that c is a positive crossing, so $n_-(D_1) = n_-(D)$. In particular, when we consider gradings, the inclusion $\mathrm{CKh}(D_1) \hookrightarrow \mathrm{CKh}(D)$ is actually an inclusion $\mathrm{CKh}^{*-1}(D_1) \hookrightarrow \mathrm{CKh}^*(D)$. On the other hand, the inclusion of $\mathrm{CKh}(D_0)$ preserves grading, so that we have

$$\mathrm{CKh}^*(D) = \mathrm{CKh}^*(D_0) \oplus \mathrm{CKh}^{*-1}(D_1).$$

Another way to think about this is that $\mathrm{CKh}(D)$ is the mapping cone of the (grading-preserving) map $\partial : \mathrm{CKh}(D_0) \to \mathrm{CKh}(D_1)$ which "forgets" the circle in D_0. In particular, we know that $\mathcal{F}(D_v \amalg S^1) =$

$\mathcal{F}(D_v) \oplus \mathcal{F}(S^1)$. Let v be some smoothing of D_0, and hence of D_1, where we order the crossings of D_0 and D_1 the same way. Then we have $\mathcal{F}((D_0)_v) = \mathcal{F}((D_1)_v) \oplus \mathcal{F}(S^1)$; the map ∂ is thus the projection to the first coordinate $\mathcal{F}((D_1)_v)$.

To define F, we must define two maps $F_0 : \mathrm{CKh}(D') \to \mathrm{CKh}(D_0)$ and $F_1 : \mathrm{CKh}(D') \to \mathrm{CKh}(D_1)$. We will define $F_1 = 0$. To define F_0, on the other hand, we will find cobordisms between the smoothings of D' to the smoothings of D_0 and then to apply the TQFT $\mathcal{F}$ to these cobordisms.

Notice that each smoothing of D' uniquely corresponds to a smoothing of D_0, namely the smoothing which resolves each crossing in the same way. The only difference in the smoothings is that D_0 has an extra copy of S^1. Thus we may define a cobordism A to be the identity everywhere, and then to be a punctured torus whose boundary is the extra loop in D_0. If one considers the small neighborhoods of D' and D_0 depicted in Figures 5 and 6, respectively, then A is the product cobordism outside of these neighborhoods. Inside the neighborhoods, it is the tube shown on the left side of Figure 7 below. On the other hand, we

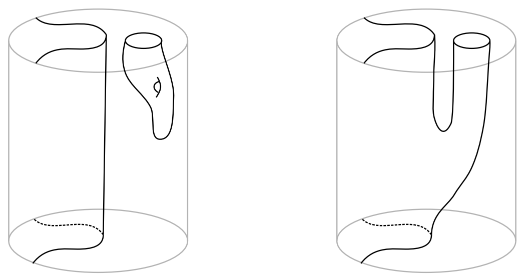

Figure 7: The two cobordisms which define F_0. (Recall that we read cobordisms from bottom to top, so these cobordisms go from D' to D_0.)

may also define a cobordism B which is again the identity outside of N. This time, however, it splits a loop in D' into the same loop (seen in D_0) and the extra copy of S^1 in D_0, as seen on the right side of Figure 7. We define F_0 to be the map

$$F_0 \coloneqq \mathcal{F}(A) - \mathcal{F}(B) : \mathrm{CKh}(D') \to \mathrm{CKh}(D_0).$$

We claim that $F = (F_0, F_1) = (\mathcal{F}(A - B), 0)$ is a chain map. Fix some smoothing v of D', say with norm i. Then dF and Fd are both obtained by applying $\mathcal{F}$ to some cobordism from v to w, where w ranges through (a subset of) smoothings of D' (or, equivalently, of D_0) with norm $i+1$. It suffices to show that dF and Fd define isomorphic cobordisms for each such w. First, notice that F does not change the smoothing in the sense that it takes an element of D'_v to an element of $(D_0)_v$. Thus the signs in dF and Fd are the same, so we may ignore them. On the other hand, notice that $S_e \circ A = A \circ S_e$, as seen in Figure 8 below. Similarly, we have $S_e \circ B = B \circ S_e$, since both cobordisms end up looking like S_e with a hole corresponding to the "arm" which connects to the extra copy of S^1 in D_0. Hence $dF_0 = F_0 d$; since we also have $F_1 = 0$, it follows that $dF = Fd$.

Now we define $G : \mathrm{CKh}(D) \to \mathrm{CKh}(D')$. We will be quite a bit more succinct here. For any $x \in G$, we may write $x = (x_0, x_1)$. We define $G(x) = G(x_0)$ to be obtained by applying the TQFT to the cobordism C in Figure 9. Once again, this is a chain map because dG and Gd both give rise to cobordisms which, from v to w, look like S_e with a punctured sphere next to it.

Now we must show that GF and FG are both homotopic to the identity. In fact, we have that GF is actually exactly *equal* to the identity on $\mathrm{CKh}(D')$. To see this, it suffices to show that the cobordisms are diffeomorphic to the identity, i.e., product, cobordism. But, using the T relation, we have the equation depicted in Figure 10 below, which proves that $GF_0 = \mathrm{id}$. Thus $GF(v) = G(F_0(v), F_1(v)) = G(F_0(v))$ is the identity, as desired.

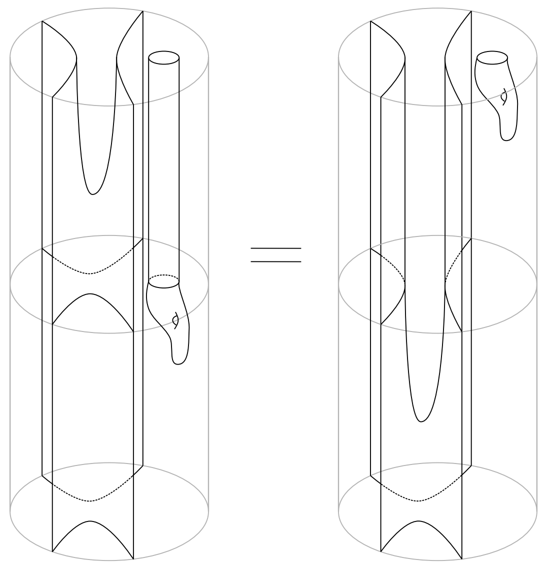

Figure 8: The cobordisms A and S_e commute since both look like S_e with a punctured torus next to it. Reading from bottom to top, the left diagram is $S_e \circ A$; the right is $A \circ S_e$.

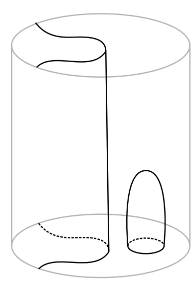

Figure 9: The cobordism which defines $G : \mathrm{CKh}(D) \to \mathrm{CKh}(D')$.

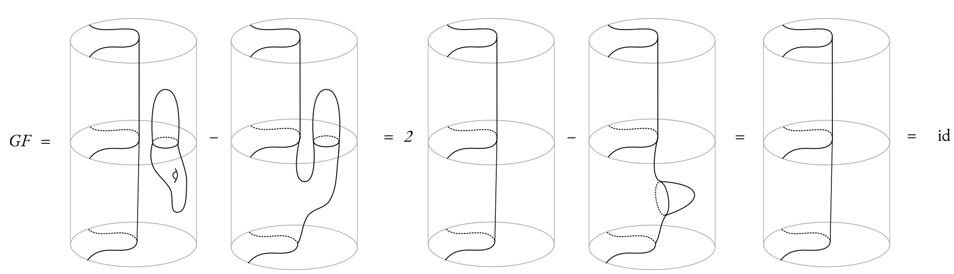

Figure 10: The composition GF is obtained by stacking the G cobordism on top of the F cobordism. The T relation implies that the first term of GF is twice the identity, and so we have $GF_0 = \mathrm{id}$.

Now we must show that FG is chain homotopic to the identity. In particular, we would like to find a family of cobordisms H_{vw} going from a smoothing v of norm i to a smoothing w of norm $i-1$ so that $\mathrm{id} - FG = Hd + dH$. Recall that c is the crossing in D which is undone by the Reidemeister move. If v and w differ at a crossing besides c, let H be the zero cobordism. Otherwise, we have $x \in \mathcal{F}((D_1)_v) \subset \mathrm{CKh}(D_1) \subset \mathrm{CKh}(D)$ and $y \in \mathcal{F}((D_0)_w) \subset \mathrm{CKh}(D_0) \subset \mathrm{CKh}(D)$, and define H as in Figure 11. Then we have $Hd(x) = 0$, so we must show that $FG - \mathrm{id} = dH$. To see this, we simply apply the $4Tu$

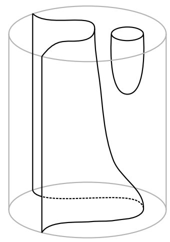

Figure 11: The cobordism h goes from D_1 to D_0.

relationship to the diagram in Figure 12. Note that C_{12} and C_{13} are the two terms in FG, while $C_{24} = \mathrm{id}$

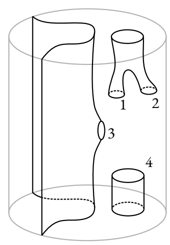

Figure 12: The four disks to which we apply the $4Tu$ relationship.

and $Hd = C_{34}$.Thus the $4Tu$ relationship completes the proof that $FG \simeq \mathrm{id}$.

Hence $\mathrm{Kh}(D)$ is invariant under the first Reidemeister move.

The proof of invariance for the R2 and R3 moves follows a similar idea, though the proofs themselves (and cobordisms involved) are more complicated. Thus we omit them here; proofs may be found in [3, 2]. □

2 Khovanov homology and variants

So far, what we have done is defined a link homology for every (1+1)-dimensional TQFT. Khovanov homology is obtained by a particular choice of TQFT $\mathcal{F}$. In particular, there is a well-known equivalence between (1+1)-dimensional TQFTs and Frobenius algebras, as mentioned briefly in class. Roughly speaking, a TQFT maps a merging cobordism to the multiplication map in the corresponding Frobenius algebra; similarly, the splitting cobordism corresponds to the comultiplication map. For details, see [1]. Thus to define a valid link homology, it suffices to present a suitable commutative Frobenius algebra over $\mathbb{Q}$ which satisfies the S, T, and $4Tu$ axioms.

The most common TQFT, and the example which is given the name of "Khovanov homology," is defined as follows. Let V be a graded two-dimensional $\mathbb{Q}$-vector space spanned by $\mathbf{v}_+$ and $\mathbf{v}_-$; we ask that $\mathbf{v}_+$ has grading +1 while $\mathbf{v}_-$ has grading −1. (We ask that gradings add along tensor products. Thus $\mathbf{v}_+ \otimes \mathbf{v}_-$, for instance, has grading 0, while $\mathbf{v}_- \otimes \mathbf{v}_-$ has grading -2.) Then we define the TQFT $\mathcal{F}_{\text{Kh}}$ to take a disjoint union of k many circles to the vector space $V^{\otimes k}$. In particular, we have $\mathcal{F}_{\text{Kh}}(\emptyset) = \mathbb{Q}$ and $\mathcal{F}_{\text{Kh}}(S^1) = V$. Furthermore, we define our multiplication map $m : V \otimes V \to V$ as follows.

$$m(\mathbf{v}_+ \otimes \mathbf{v}_+) = \mathbf{v}_+, \quad m(\mathbf{v}_+ \otimes \mathbf{v}_-) = m(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_-, \quad m(\mathbf{v}_- \otimes \mathbf{v}_-) = 0$$

Furthermore, we define the comultiplication map $\Delta : V \to V \otimes V$ by

$$\Delta(\mathbf{v}_+) = \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+, \quad \Delta(\mathbf{v}_-) = \mathbf{v}_- \otimes \mathbf{v}_-.$$

Finally, we define our unit and counit maps by

$$\iota(1) = \mathbf{v}_+$$

and

$$\varepsilon(\mathbf{v}_+) = 0, \quad \varepsilon(\mathbf{v}_-) = 1,$$

respectively. (In the geometric language of TQFTs, the unit and counit maps corresponds to births and deaths of circles, i.e., to attachments of 0- and 2-handles, respectively.) One may check that this does indeed define a Frobenius algebra (or a TQFT).

Proposition 3. *The Frobenius algebra $\mathcal{F}_{\text{Kh}}$ defined above satisfies the S, T, and $4Tu$ relations. In particular, it defines a link homology, which we call* Khovanov homology *and denote* $\text{Kh}(L)$.

Proof. First, we verify the S relation. By decomposing the sphere as a 0-handle and a 2-handle, it follows that the map $\mathcal{F}_{\text{Kh}}(S^2) : \mathcal{F}_{\text{Kh}}(\emptyset) \to \mathcal{F}_{\text{Kh}}(\emptyset)$ may be written as the composition

$$\mathbb{Q} = \mathcal{F}_{\text{Kh}}(\emptyset) \xrightarrow{\iota} \mathcal{F}_{\text{Kh}}(S^1) \xrightarrow{\varepsilon} \mathcal{F}_{\text{Kh}}(\emptyset) = \mathbb{Q}.$$

The middle term is exactly $V = \mathbb{Q}\mathbf{v}_+ \oplus \mathbb{Q}\mathbf{v}_-$. This map takes $1 \mapsto \mathbf{v}_+ \mapsto 0$, so the sphere S^2 corresponds to the number 0. This is exactly the S relation.

For the T relation, we decompose the torus as a 0-handle, followed by a split, then a merge, and finally a 2-handle, as seen in Figure 13. Then we have that $\mathcal{F}_{\text{Kh}}(T^2)$ may be written as the composition

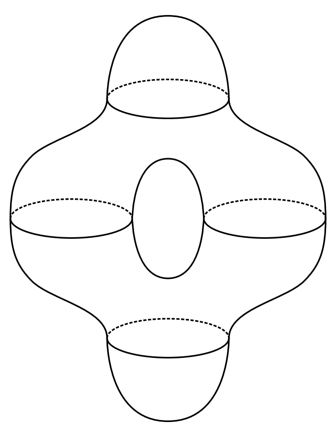

Figure 13: The torus may be decomposed into the birth of a circle, followed by a split and a merge, and finally the death of a circle.

$$\mathbb{Q} \xrightarrow{\iota} V \xrightarrow{\Delta} V \otimes V \xrightarrow{m} V \xrightarrow{\varepsilon} \mathbb{Q}.$$

In particular, this maps takes $1 \mapsto \mathbf{v}_+ \mapsto \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- + \mathbf{v}_+ \mapsto 2\mathbf{v}_- \mapsto 2$. Hence the torus corresponds to $2 \in \mathbb{Q}$, which is exactly the T relation.

Finally, we must verify the $4Tu$ relation. First, observe that we may ask that the circular boundaries of the four disks lie in the same horizontal "slice" of the larger cobordism, drawn as a potentially disjoint union of cylinders. We may also ask that the interiors of these disks lie below this slice. The map associated to the four different cobordisms in the $4Tu$ relation is the same above the slice; furthermore, below the slice, the only difference is in this small neighborhood where the tubes are created. Thus the local picture we must show is that Figure 14.

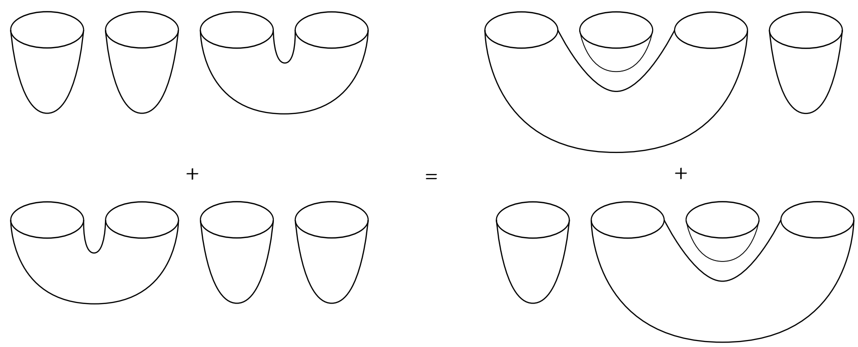

Figure 14: The local picture we would like to show in order to prove that $\mathcal{F}_{\mathrm{Kh}}$ satisfies the $4Tu$ relation.

On the left side, we have the map

$$\Delta \circ \iota \otimes \iota \otimes \iota + \iota \otimes \iota \otimes \Delta \circ \iota : \mathcal{F}_{\mathrm{Kh}}(\emptyset) \to \mathcal{F}_{\mathrm{Kh}}(V^{\otimes 4}).$$

The first term takes 1 to $\mathbf{v}_+ \otimes \mathbf{v}_- \otimes \mathbf{v}_+ \otimes \mathbf{v}_+ + \mathbf{v}_- \otimes \mathbf{v}_+ \otimes \mathbf{v}_+ \otimes \mathbf{v}_+$; the second takes 1 to $\mathbf{v}_+ \otimes \mathbf{v}_+ \otimes \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_+ \otimes \mathbf{v}_+ \otimes \mathbf{v}_- \otimes \mathbf{v}_+$. On the right side, a similar argument implies that the local picture corresponding to C_{13} maps 1 to $\mathbf{v}_+ \otimes \mathbf{v}_+ \otimes \mathbf{v}_- \otimes \mathbf{v}_+ + \mathbf{v}_- \otimes \mathbf{v}_+ \otimes \mathbf{v}_+ \otimes \mathbf{v}_+$. (Note that the map in this case is $p_1 \Delta \iota \otimes \iota \otimes p_2 \Delta \otimes \iota$, where $p_1, p_2 : V \otimes V \to V$ are the projections to the first and second coordinates, respectively.) Similarly, the local picture corresponding to C_{24} takes 1 to $\mathbf{v}_+ \otimes \mathbf{v}_+ \otimes \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_+ \otimes \mathbf{v}_- \otimes \mathbf{v}_+ \otimes \mathbf{v}_+$. Thus both $C_{12} + C_{34}$ and $C_{13} + C_{24}$ take 1 to the sum of the four terms with exactly one $\mathbf{v}_-$ and three $\mathbf{v}_+$'s. In particular, these are equal, so $\mathcal{F}_{\mathrm{Kh}}$ satisfies the $4Tu$ relation, as desired. □

In fact, because the base vector space V was graded, Khovanov homology is actually a bigraded theory. On one hand, there is the homological grading from before; on the other hand, there is a new grading, known as the *quantum grading*, coming from the grading of V. In particular, if $x \in \mathcal{F}(D_v)$ is a homogeneous element in $\mathrm{CKh}(D)$, then we define $p(x)$ to be the grading of x as an element of $V \otimes \cdots \otimes V$. Then we define the quantum degree to be

$$\mathrm{qdeg}(x) := p(x) + \mathrm{gr}(v) + n_+ - n_- = p(x) + |v| + n_+ - 2n_-.$$

Here n_+ and n_- denote the number of positive and negative crossings in D. They are included in the computation of the degree so that $\mathrm{qdeg}(x)$ is independent of the diagram.

A variant of this theory, due to Eun Soo Lee [4], is defined by slightly different cobordism maps. In particular, everything is the same except $m(\mathbf{v}_- \otimes \mathbf{v}_-) = \mathbf{v}_+$ and $\Delta(\mathbf{v}_-) = \mathbf{v}_+ \otimes \mathbf{v}_+ + \mathbf{v}_- \otimes \mathbf{v}_-$. One may verify that this also satisfies the three necessary relations, so this too gives a link homology, often known as *Lee homology*. We denote it by $\mathrm{Lee}(L)$. Because Δ, for example, is not even homogeneous, we do not get a bigraded theory. Indeed, the quantum grading actually defines a filtration on the complex $\mathrm{CKh}(L)$; for more details see [5].

Finally, to illustrate these two theories, we present a computation of the homologies of the Hopf link L in Figure 15. The cube of resolutions for the diagram D of L in Figure 15 is as shown in Figure 16.

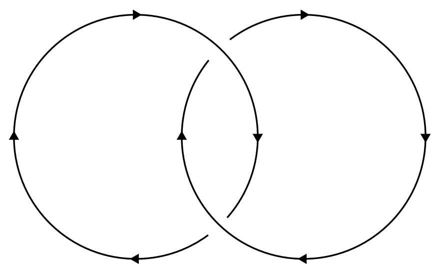

Figure 15: A diagram of the Hopf link with two positive crossings. (Orientations will be important for determining grading, which depends on the number of positive and negative crossings.)

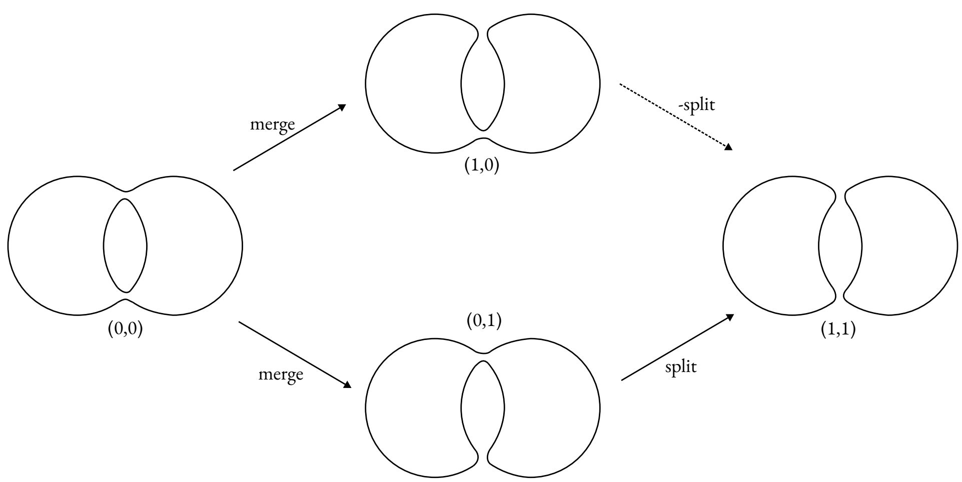

Figure 16: The cube of resolutions for the Hopf link. Each column corresponds to smoothings of fixed grading; since $n_+ = 2$ and $n_- = 0$, the grading is the same as the norm of the smoothing, so the columns corresponds to homological gradings 0, 1, and 2, from left to right. The maps are given by merges and splits, as indicated. The dashed arrow indicates a negative arrow.

In particular, the chain complex $\mathrm{CKh}(D)$ is given by

$$\mathcal{F}(S^1 \amalg S^1) \to \mathcal{F}(S^1) \oplus \mathcal{F}(S^1) \to \mathcal{F}(S^1).$$

Specifying the homological gradings, these terms correspond to $\mathrm{CKh}^0(D)$, $\mathrm{CKh}^1(D)$, and $\mathrm{CKh}^2(D)$ from left to right. Since Khovanov and Lee homology have the same chain complexes, this complex is always

$$V \otimes V \to V \oplus V \to V.$$

The first map sends $x \otimes y \mapsto (m(x \otimes y), m(x \otimes y))$, while the second map sends $x \oplus y$, where x belongs to the $(1,0)$-smoothing and y to the $(0,1)$-smoothing, to $-\Delta(x) + \Delta(y)$.

We compute the Khovanov homology first. Then the first differential $d^0 : \mathrm{CKh}^0(D) \to \mathrm{CKh}^1(D)$ is defined by

$$d^0(\mathbf{v}_+ \otimes \mathbf{v}_+) = (\mathbf{v}_+, \mathbf{v}_+), \quad d^0(\mathbf{v}_+ \otimes \mathbf{v}_-) = d^0(\mathbf{v}_- \otimes \mathbf{v}_+) = (\mathbf{v}_-, \mathbf{v}_-), \quad d^0(\mathbf{v}_- \otimes \mathbf{v}_-) = (0, 0).$$

This has two-dimensional kernel spanned by $\mathbf{v}_- \otimes \mathbf{v}_-$ and $\mathbf{v}_+ \otimes \mathbf{v}_- - \mathbf{v}_- \otimes \mathbf{v}_+$, and so it follows that $\mathrm{Kh}^0(L) = \mathbb{Q}^2$. Furthermore, the quantum grading of the two elements are 0 and 2, respectively, so it follows that $\mathrm{Kh}^{0,0}(L)$ and $\mathrm{Kh}^{0,2}(L)$ are both $\mathbb{Q}$.

The image of d^0, on the other hand, is the two-dimensional space (x, x). Now d^1 is defined by

$$-d^1(\mathbf{v}_+, 0) = \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ = d^1(0, \mathbf{v}_+), \quad -d^1(\mathbf{v}_-, 0) = \mathbf{v}_- \otimes \mathbf{v}_- = d^1(0, \mathbf{v}_-).$$

Hence $(\mathbf{v}_+, \mathbf{v}_+)$ and $(\mathbf{v}_-, \mathbf{v}_-)$ span the kernel of d^1. Thus $\mathrm{Kh}^1(L) = \ker d^1 / \operatorname{im} d^0$ is 0.

Finally, the image of d^1 is spanned by $\mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+$ and $\mathbf{v}_- \otimes \mathbf{v}_-$. On the other hand, the kernel of d^2 contains everything, so $\ker d^2 / \operatorname{im} d^1$ is spanned by $\mathbf{v}_+ \otimes \mathbf{v}_+$ and $\mathbf{v}_+ \otimes \mathbf{v}_-$. The former has quantum degree 6 while the latter has quantum degree 4.

Hence we have the following calculation:

$$\mathrm{Kh}^{i,j}(L) = \begin{cases} \mathbb{Q} & \text{if } (i, j) = (0, 0), (0, 2), (2, 4), \text{ or } (2, 6), \\ 0 & \text{otherwise.} \end{cases}$$

Now we compute the Lee homology. We compute the differentials d^0 and d^1 again:

$$d^0(\mathbf{v}_+ \otimes \mathbf{v}_+) = (\mathbf{v}_+, \mathbf{v}_+), \quad d^0(\mathbf{v}_+ \otimes \mathbf{v}_-) = d^0(\mathbf{v}_- \otimes \mathbf{v}_+) = (\mathbf{v}_-, \mathbf{v}_-), \quad d^0(\mathbf{v}_- \otimes \mathbf{v}_-) = (\mathbf{v}_+, \mathbf{v}_+)$$

$$-d^1(\mathbf{v}_+, 0) = \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ = d^1(0, \mathbf{v}_+), \quad -d^1(\mathbf{v}_-, 0) = \mathbf{v}_+ \otimes \mathbf{v}_+ + \mathbf{v}_- \otimes \mathbf{v}_- = d^1(0, \mathbf{v}_-).$$

Thus $\ker d^0$ is spanned by $\mathbf{v}_+ \otimes \mathbf{v}_+ - \mathbf{v}_- \otimes \mathbf{v}_-$ and $\mathbf{v}_+ \otimes \mathbf{v}_- - \mathbf{v}_- \otimes \mathbf{v}_+$. Hence $\mathrm{Lee}^0(L) = \mathbb{Q}^2$. On the other hand, we know that $\operatorname{im} d^0$ and $\ker d^1$ are again both spanned by $\mathbf{v}_+ \otimes \mathbf{v}_+$ and $\mathbf{v}_- \otimes \mathbf{v}_-$, so that $\mathrm{Lee}^1(L) = 0$. Finally, we have that $\operatorname{im} d^1$ is spanned by $\mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+$ and $\mathbf{v}_+ \otimes \mathbf{v}_+ + \mathbf{v}_- \otimes \mathbf{v}_-$, so that $\mathrm{Lee}^2(L) = \mathbb{Q}^2$ is spanned by $\mathbf{v}_+ \otimes \mathbf{v}_+$ and $\mathbf{v}_+ \otimes \mathbf{v}_-$.

In this case, when considered purely as (ungraded) vector spaces, we have $\mathrm{Kh}(L) \cong \mathrm{Lee}(L) \cong \mathbb{Q}^4$. However, this is not generally true. For example, one may compute that the Khovanov homology of the trefoil is four-dimensional, while its Lee homology is only two-dimensional. It turns out, however, that $\operatorname{rank} \mathrm{Lee}(L) \leq \operatorname{rank} \mathrm{Kh}(L)$ in general. While we do not describe the details here, this may be proved by the fact that quantum grading defines a filtration on the Lee chain complex; one may eventually use this filtration to obtain a spectral sequence from Khovanov homology to Lee homology, as in [5]. Furthermore, Lee proved in [4] that the rank of Lee homology is always equal to $2^{|L|}$, where $|L|$ is the number of components in L, this spectral sequence gives us useful information about Khovanov homology.

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