A History of the Arnol'd Conjecture

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The Arnol'd conjecture, roughly speaking, bounds the number of 1-periodic trajectories of a certain kind of vector field on a symplectic manifold M by a purely topological quantity, namely $\sum \dim H_k(M; \mathbb{Z}/2)$. It comes from a generalization of Poincaré's geometric theorem, which he discovered in 1912 while studying periodic solutions to certain problems in celestial mechanics.

In particular, Poincaré was studying a version of the *three-body problem.* In this problem, two bodies orbit in a circle around their center of mass. A third, significantly smaller body is introduced to this system and moves in some orbit around the two existing bodies. This could be, for example, a satellite which enters the Earth–Moon system. Poincaré wanted to show that this third body could have periodic orbits.

To do so, he came up with a simple model for these problems, namely an area-preserving map from the annulus to itself. He showed that there was a bijection between the periodic solutions he wanted to find and fixed points of a corresponding area-preserving homeomorphism of the annulus. By studying the fixed points of such maps, he discovered the following theorem, which George Birkhoff proved the following year.

Theorem 1 (Poincaré's geometric theorem)**.** *Every area-preserving homeomorphism of the annulus $S^1 \times [-1, 1]$ which rotates the boundary circles in opposite directions has at least two fixed points.*

Given this theorem, it is natural to attempt to find a generalization that gives a lower bound on the number of fixed points of some collection of maps with a given property.

As a first step in generalizing Poincaré's geometric theorem, we can consider maps on the torus. After all, we can decompose the torus into two circular annuli which are joined by two connecting annuli, as shown in Figure 1.

Intuitively, then, Poincaré's geometric theorem implies that a suitable diffeomorphism of the torus will have at least four fixed points. This is indeed the case, subject to certain conditions. In particular, the following proposition is true.

Proposition 2. *Consider a symplectomorphism*

$$(x, y) \mapsto (x + f(x, y), y + g(x, y)) = (X, Y)$$

which fixes the center of gravity in the sense that the average values of f and g are zero as x and y range over S^1. This map has at least four fixed points, with multiplicity.

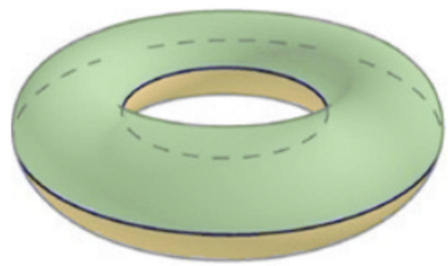

Figure 1: The torus $S^1 \times S^1$ can be constructed as two annuli connected by a thin annulus (indicated by the equatorial lines) on each boundary circle.

In particular, given any area-preserving homeomorphism of the annulus, we can extend it to a diffeomorphism on the torus which preserves the center of gravity. This diffeomorphism is equal to the original homeomorphism on each of the two main annuli. It translates the two connecting annuli, but in opposite directions. By choosing the magnitude of the translations, we can ensure that the diffeomorphism preserves the center of gravity.

If we add an eigenvalue condition to Proposition 2, namely that -1 is not an eigenvalue of the Jacobian at any point of the torus, then this follows from a correspondence between fixed points of a *symplectomorphism*, which we define below, and critical points of a corresponding function known as a *Hamiltonian*. Without this eigenvalue condition, the proof runs into the same difficulties as the original proof of Poincaré's geometric theorem.

Proposition 2 uses a few definitions from symplectic geometry. In particular, a *(smooth) manifold* M is a space which locally looks like Euclidean space $\mathbb{R}^n$. It can be thought of as a higher-dimensional analogue of curves (1-dimensional manifolds) and surfaces (2-dimensional manifolds). A *symplectic manifold* is a manifold M equipped with an additional structure, namely a *symplectic structure*. A symplectic structure is a closed 2-form ω such that ω_x is *nondegenerate* at every $x \in M$ in the sense that, for every nonzero tangent vector $X \in T_xM$, there is a Y with $\omega_x(X, Y) \neq 0$.

As one example of a symplectic manifold, consider the Euclidean space $\mathbb{R}^{2n}$ with coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$. Then the 2-form $\sum dq_i \wedge dp_i$ makes $\mathbb{R}^{2n}$ into a symplectic manifold.

A map $f : M_1 \to M_2$ between two manifolds is a *diffeomorphism* if it is a bijective smooth (i.e., infinitely differentiable) map whose inverse is also smooth. The identity map on any smooth manifold is trivially a diffeomorphism. A *symplectomorphism* between the symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is simply a diffeomorphism $f : M_1 \to M_2$ which preserves the symplectic structures in the sense that $f^*\omega_2 = \omega_1$. More explicitly, this means that

$$(\omega_2)_{f(x)}(df_x(X), df_x(Y)) = (\omega_1)_x(X, Y)$$

for every $x \in M_1$ and $X, Y \in T_xM_1$.

Furthermore, it will turn out to be helpful to have the definition of a Hamiltonian vector field. In particular, if (M, ω) is a symplectic manifold and $H : M \to \mathbb{R}$ is a function, then we can form the *Hamiltonian vector field* X_H as the vector field satisfying the condition that

$$\omega_x(Y, X_H(x)) = dH_x(Y)$$

for every $x \in M$ and $Y \in T_xM$. In the *time-dependent* case, we let $H : M \times \mathbb{R} \to \mathbb{R}$ be a function such that $X_t = X_{H_t}$, where $H_t(x) = H(x, t)$.

It turns out that Proposition 2 does, as a matter of fact, lead to the correct generalization of Poincaré's geometric theorem. As such, we must now find a way to reformulate the condition that the symplectomorphism "preserves the center of gravity." The appropriate reformulation is that of being "exactly homotopic to the identity," which, in the case of the torus, is equivalent to the notion of preserving the center of gravity.

In particular, let $g : M \to M$ be a symplectomorphism. Consider a smooth time-dependent function $H : M \times \mathbb{R} \to \mathbb{R}$ which is 1-periodic in the sense that $H(x, t) = H(x, t + 1)$ for all $(x, t) \in M \times \mathbb{R}$. Then we say that g is *generated by* H if g is equal to the time-1 flow of X_t. In other words, letting ψ be the flow of X_t, so that $\psi^0 = \mathrm{id}_M$ and

$$\frac{d}{dt}\psi^t = X_t(\psi^t),$$

we say that g is generated by H if then $g = \psi^1$. If such an H exists, then we say that g is *exactly homotopic to the identity*.

With all this new terminology, we can rephrase Proposition 2 as follows.

Proposition 2'. *Let $H : T \times \mathbb{R} \to \mathbb{R}$ be a 1-periodic time-dependent Hamiltonian on the torus. If g is generated by H, then it has at least four fixed points.*

This proposition is generalized by Arnol'd's conjecture, which gives a lower bound on the number of fixed points that a symplectomorphism g on *any* manifold can have, assuming that g is exactly homotopic to the identity. Notice that a fixed point of $g = \psi^1$ corresponds to a periodic solution of period 1 of the Hamiltonian system, so it is equivalent to give a lower bound on the number of 1-periodic solutions to a Hamiltonian system.

In the simpler case that the function H is time-independent, we can find such a lower bound. In particular, we can show that any critical point of $H : M \to \mathbb{R}$ is a 1-periodic solution, so that g has at least as many fixed points as H has critical points.

After all, if x is a critical point of H, then $dH_x = 0$ by definition. Let X_H is the Hamiltonian vector field associated with H. Now the nondegeneracy of ω and the fact that

$$\omega_x(Y, X_H(x)) = dH_x(Y) = 0$$

for all $Y \in T_xM$ together imply that $X_H(x) = 0$. This is true if and only if x lies on a constant trajectory. Since this trajectory is obviously periodic of any period, including of period 1, it follows that x corresponds to a periodic solution of period 1, as desired. (And, in fact, it corresponds to a fixed point of the symplectomorphism g given by ψ^1, where ψ is the flow of X_H.)

Thus we have shown the following.

Proposition 3. *If $H : M \to \mathbb{R}$ is a time-independent Hamiltonian and g is the time-1 flow of X_H, then*

$$\# \text{ fixed points of } g \geq \# \text{ critical points of } H.$$

In the time-dependent case, the corresponding result to this proposition is unsolved. This is the Morse theoretic version of Arnol'd's conjecture. Before we state the conjecture, we must define a Morse function. In particular, a *Morse function* $f : M \to \mathbb{R}$ on M is a real-valued function whose critical points are nondegenerate in the sense that the determinant of the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is nonzero.

Conjecture 4 (Morse theoretic Arnol'd conjecture)**.** *Let (M, ω) be a compact symplectic manifold and $H : M \times \mathbb{R} \to \mathbb{R}$ be a time-dependent Hamiltonian. Let $X_t = X_{H_t}$ be the (time-dependent) Hamiltonian vector field associated to H. Suppose now that the 1-periodic solutions of the associated Hamiltonian system*

$$\dot{x}(t) = X_t(x(t))$$

are nondegenerate in the sense that $d\psi^1$ does not have eigenvalue 1 at x. Then the number of such 1-periodic solutions is at least the minimal number of critical points which a Morse function on M can have.

The two ideas of nondegeneracy—first, as a critical point of the function H, and second, as a periodic solution to the Hamiltonian system defined by H—are, in fact, related. In particular, if a point x is nondegenerate in the second sense, then it is nondegenerate in the first sense. This can be shown via an explicit calculation involving Darboux's theorem, which states that every symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \sum dq_i \wedge dp_i)$, and the order 2 Taylor expansion of H.

It turns out that, given a Morse function f on a manifold M, we can construct the *Morse complex.* In particular, let $C_k = \mathbb{Z}/2\{\mathrm{Crit}_k(f)\}$, where $\mathrm{Crit}_k(f)$ is the set of critical points of f with index k. Then one can define a differential $\partial : C_k \to C_{k-1}$ making (C_*, ∂) a complex, known as the Morse complex. In fact, one can show that the homology of the Morse complex doesn't depend on which Morse function f is picked and is the same as cellular homology.

From this, it follows that the number of critical points of a Morse function on M is bounded below by the sum $\sum \dim H_k(M; \mathbb{Z}/2)$. After all, we know that $H_k(M; \mathbb{Z}/2)$ is a subquotient of C_k, and therefore has dimension at most $\dim C_k = \#\mathrm{Crit}_k(f)$. In fact, this leads to the statement of the homological Arnol'd conjecture, which is weaker than the Morse theoretic Arnol'd conjecture and which has, in fact, been proven.

Theorem 5 (Homological Arnol'd conjecture)**.** *With the hypotheses from Conjecture 4, the number of 1-periodic solutions to the Hamiltonian system defined by H is greater than or equal to the sum*

$$\sum_k \dim H_k(M; \mathbb{Z}/2).$$

In the case of the torus, this is equivalent to Proposition 2. After all, the homology groups of a torus are $H_0(T; \mathbb{Z}/2) = \mathbb{Z}/2 = H_2(T; \mathbb{Z}/2)$ and $H_1(T; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Thus the sum of their dimensions is exactly 4, so Theorem 5 implies that the number of fixed points of g, which is equal to the number of 1-periodic solutions of the associated Hamiltonian system H ,is at least $\sum \dim H_k(T; \mathbb{Z}/2) = 4$.

The proof of the (homological) Arnol'd conjecture uses Floer theory. In particular, on a certain infinite-dimensional space, we can define a the *action functional* $\mathcal{A}_H$ whose critical points are exactly the 1-periodic solutions to the Hamiltonian system defined by H. Just as how the critical points of a Morse function on M give rise to a chain complex, so too do the critical points of $\mathcal{A}_H$ allow us to construct an associated chain complex. (The hard part of this proof is defining the differential and proving that it gives a complex.)

It turns out that the homology of this complex, known as the Floer homology, is exactly equal to the Morse, and hence cellular, homology. Hence the number of 1-periodic solutions is equal to the number of critical points of $\mathcal{A}_H$, which in turn is at least $\sum \dim H_k(M; \mathbb{Z}/2)$, as desired.

References

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